

# THEORY OF CYCLOSTATIONARY POPS

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## 1. INTRODUCTION

In this manuscript, the formal concept of the "cyclostationary POPs" is presented. Cyclostationary POPs, which are generalization of ordinary (stationary) POPs (Hasselmann, 1988; Storch et al., 1988), have been suggested originally by Hasselmann (1985?; unpubl. manuscript), and have been designed by Ortiz et al (1990) and Blumenthal (1990).

It should be noted that the following formalism may be applied to linear systems whose system matrices are estimated from data or whose system matrices are derived from theoretical dynamical considerations (e.g., Kelvin and Rossby waves in the equatorial Pacific whose periods are comparable with the annual cycle).

The following notations are used: Vectors are given in **bold** and matrices are given in gothic. If  $\mathfrak{A}$  is a matrix then  $\mathfrak{A}^T$  is the transposed matrix. If  $x$  is any complex quantity then  $x^*$  is the conjugate complex. The time is given by a pair of integers  $(t, \tau)$ , with  $t$  counting the cycles (annual cycle, diurnal cycle, QBO as an external cycle), and  $\tau$  counting the "seasonal date", i.e., the time steps within a cycle, with  $\tau = 1, \dots, n$ . It is  $(t, n+1) = (t+1, 1)$  or, generally,  $(t, \tau+n) = (t+1, \tau)$ . Then the cyclostationary process may be written as

$$(1) \quad \mathbf{y}(t, \tau+1) = \mathfrak{A}(\tau) \cdot \mathbf{y}(t, \tau) + \text{noise}$$

with  $\mathbf{y}(t, \tau+n) = \mathbf{y}(t+1, \tau) \quad \text{and } \mathfrak{A}(\tau+n) = \mathfrak{A}(\tau)$

## 2) THE t-CYCLE POP MODELS

### a) The eigenvectors

Applying (1) consecutively n-times we find with

$$(2) \quad \mathcal{B}(\tau) = \mathcal{A}(\tau+n-1) \cdot \mathcal{A}(\tau+n-2) \cdots \mathcal{A}(\tau+1) \cdot \mathcal{A}(\tau)$$

that

$$(3) \quad \mathbf{y}(t+1, \tau) = \mathcal{B}(\tau) \cdot \mathbf{y}(t, \tau) + \text{noise}$$

Because of the imposed periodicity n of the models (3), named here t-cycle POP models, exist. To each of these a conventional POP analysis can be applied. In this way eigenvectors  $\mathbf{p}_j^\tau$  and eigenvalues  $\lambda_j^\tau$  are obtained:

$$\mathcal{B}(\tau) \mathbf{p}_j^\tau = \lambda_j^\tau \mathbf{p}_j^\tau \quad \text{with } \mathbf{p}_j^{\tau T^*} \cdot \mathbf{p}_j^\tau = 1$$

In the following the sub-index j is dropped, for convenience. It is easily shown that the eigenvalues  $\lambda^\tau$  are the same for different t-cycle POP models:

$$\mathcal{B}(\tau) \mathbf{p}^\tau = \lambda^\tau \mathbf{p}^\tau \Rightarrow \mathcal{A}(\tau+n) \cdot \mathcal{B}(\tau) \mathbf{p}^\tau = \lambda^\tau \mathcal{A}(\tau+n) \cdot \mathbf{p}^\tau$$

$$\text{Since } \mathcal{A}(\tau+n) = \mathcal{A}(\tau), \text{ using (2): } \Rightarrow \mathcal{B}(\tau+1) \cdot [\mathcal{A}(\tau) \mathbf{p}^\tau] = \lambda^\tau [\mathcal{A}(\tau) \mathbf{p}^\tau].$$

Thus  $\mathcal{B}(\tau+1)$  and  $\mathcal{B}(\tau)$  share the same eigenvalues, and  $\mathcal{A}(\tau) \cdot \mathbf{p}^\tau$  is an eigenvector of  $\mathcal{B}(\tau+1)$  if  $\mathbf{p}^\tau$  is an eigenvector of  $\mathcal{B}(\tau)$ . Of course, the eigenvectors are subject to normalization, i.e.,

$$(4) \quad \mathbf{p}^{\tau+1} = \left( r_\tau \cdot e^{i\phi_\tau} \right) \mathcal{A}(\tau) \cdot \mathbf{p}^\tau$$

with a real constant  $r_\tau$  chosen so that  $\mathbf{p}^{\tau T^*} \cdot \mathbf{p}^\tau = 1$ . If, for a certain  $\tau$ , the normalization condition  $\mathbf{p}^{\tau T^*} \cdot \mathbf{p}^\tau = 1$  is fulfilled, then

$$\left( \mathbf{p}^{\tau+1} \right)^T \cdot \mathbf{p}^{\tau+1} = \left( r_\tau \cdot e^{-i\phi_\tau} \right) \cdot \left( r_\tau \cdot e^{i\phi_\tau} \right) \cdot \mathbf{p}^{\tau T^*} \mathcal{A}(\tau)^T \cdot \mathcal{A}(\tau) \cdot \mathbf{p}^\tau$$

or

$$(5) \quad r_\tau = \| \mathcal{A}(\tau) \cdot \mathbf{p}^\tau \|^{-1}$$

The angle  $\phi_\tau$  is chosen so that  $\mathbf{p}^{\tau+n} = \mathbf{p}^\tau$ . Repeated application of (4) yields:

$$\begin{aligned}\mathbf{p}^{\tau+n} &= \left( r_{\tau+n-1} \cdot e^{i\phi_{\tau+n-1}} \right) \mathbf{a}(\tau+n-1) \cdot \mathbf{p}^{\tau+n-1} \\ &= \left( \prod_{j=0}^{n-1} r_{\tau+j} \cdot \prod_{j=0}^{n-1} e^{i\phi_{\tau+j}} \right) \left[ \prod_{j=0}^{n-1} \mathbf{a}(\tau+j) \right] \cdot \mathbf{p}^\tau \\ &= \left( \prod_{j=0}^{n-1} r_{\tau+j} \cdot \prod_{j=0}^{n-1} e^{i\phi_{\tau+j}} \right) \mathbf{B}(\tau) \cdot \mathbf{p}^\tau = \lambda^\tau \cdot \left( \prod_{j=0}^{n-1} r_{\tau+j} \cdot \prod_{j=0}^{n-1} e^{i\phi_{\tau+j}} \right) \mathbf{p}^\tau\end{aligned}$$

so that  $(\lambda^\tau)^{-1} = \prod_{j=0}^{n-1} r_{\tau+j} \cdot \exp\left[i \sum_{j=0}^{n-1} \phi_{\tau+j}\right]$ , or, with  $\lambda = \rho e^{i\psi}$ :

$$(6) \quad \rho = \left[ \prod_{j=0}^{n-1} r_{\tau+j} \right]^{-1} \quad \text{and } -\psi = \sum_{j=0}^{n-1} \phi_{\tau+j}$$

We set

$$(7) \quad \phi_\tau = -\psi/n \quad \text{for all } \tau$$

Relation (6) and (7) are reasonable: Within one  $\tau$ -cycle, the POP is damped by the factor  $\rho$  and rotated by an angle  $\psi$ . Thus, to ensure  $\mathbf{p}^{\tau+n} = \mathbf{p}^\tau$ , at each time step the pattern is amplified by  $r_\tau$  and rotated backwards by  $-\psi/n$ .

So far, we have described through (4) how the different POPs,  $\mathbf{p}^1, \dots, \mathbf{p}^n$ , in a cycle are related to each other, and have imposed by means of (5) the geometric constraints  $\mathbf{p}_j^{\tau+1} \cdot \mathbf{p}_j^\tau = 1$ . The cycle is, however, not yet uniquely determined: The patterns  $\tilde{\mathbf{p}}^1, \dots, \tilde{\mathbf{p}}^n$  with  $\tilde{\mathbf{p}}^\tau = e^{i\mu} \mathbf{p}^\tau$ , with any real number  $\mu$ , transform into each other as the cycle  $\mathbf{p}^1, \dots, \mathbf{p}^n$  does, and satisfies the geometric constraint also. To make the cycle unique, we are choosing that pattern  $\mathbf{p}^1$  whose real part is closest to a positive constant, i.e., that is minimizing  $\|Re(\mathbf{p}^1)-1\|$ . In the examples to come, we present cyclostationary POPs and (conventional) stationary POPs at the same time. To allow for a better comparison, the selection for both sets of patterns, via the minimization of  $\|Re(\mathbf{p}^1)-1\|$ , is the same.

### b) The time coefficients

Usually the set of eigenvectors  $\mathbf{p}^\tau$  is at each seasonal date  $\tau$  (linearly) complete so that for any observed field  $\mathbf{y}(t, \tau)$  there is an unique expansion

$$(8) \quad \mathbf{y}(t, \tau) = \sum_j z_j(t, \tau) \mathbf{p}_j^\tau$$

Then, using (4) and (1)

$$\begin{aligned} \sum_j z_j(t, \tau+1) \mathbf{p}_j^{\tau+1} &= \mathbf{y}(t, \tau+1) = \mathbf{z}(\tau) \mathbf{y}(t, \tau) + \text{noise} \\ &= \sum_j z_j(t, \tau) \mathbf{z}(\tau) \mathbf{p}_j^\tau + \text{noise} \\ &= \sum_j z_j(t, \tau) \left( r_\tau e^{-i\psi/n} \right)^{-1} \left( r_\tau e^{-i\psi/n} \right) \cdot \mathbf{z}(\tau) \mathbf{p}_j^\tau + \text{noise} \\ &= \sum_j z_j(t, \tau) \left( r_\tau e^{-i\psi/n} \right)^{-1} \mathbf{p}_j^{\tau+1} + \text{noise} \end{aligned}$$

or,

$$(9) \quad z_j(t, \tau+1) = \left( r_\tau^{-1} e^{i\psi/n} \right) \cdot z_j(t, \tau) + \text{noise} \quad \text{for all } j.$$

In the following the sub-index  $j$  is dropped again. Repeated application of (9) yields, not unexpectedly, the conventional POP model result:

$$\begin{aligned} z(t+1, \tau) &= \prod_{k=0}^{n-1} \left( r_{\tau+k}^{-1} e^{i\psi/n} \right) \cdot z(t, \tau) + \text{noise} \\ &= \left( \prod_{k=0}^{n-1} r_{\tau+k} \right)^{-1} e^{i\psi} \cdot z(t, \tau) + \text{noise} = \lambda \cdot z(t, \tau) + \text{noise} \end{aligned}$$

There are two ways of determining the POP coefficients  $t(t, \tau)$ . The more elegant approach is using the "adjoint patterns", described in the next subsection. A straight-forward, and fairly robust, way is to fit at each time the POP patterns to the full field, i.e., by minimizing

$$(9a) \quad \| \mathbf{y}(t, \tau) - z(t, \tau) \cdot \mathbf{p}^\tau - [z(t, \tau) \cdot \mathbf{p}^\tau]^* \| \quad \text{if } \mathbf{p}^\tau \text{ is complex, or}$$

$$(9b) \quad \| \mathbf{y}(t, \tau) - z(t, \tau) \cdot \mathbf{p}^\tau \| \quad \text{if } \mathbf{p}^\tau \text{ is real}$$

The solution of (9a) is, with the notation  $z = z_1 + iz_2$  and  $\mathbf{p} = \mathbf{p}_1 + i\mathbf{p}_2$  with real numbers  $z_1, z_2$  and real vectors  $\mathbf{p}_1, \mathbf{p}_2$ :

$$\begin{aligned} \sum_t (\mathbf{y}(t) - z_1 \mathbf{p}_1 - z_2 \mathbf{p}_2)^2 &= \mathbf{y}^2 - 2z_1 \mathbf{p}_1 \mathbf{y} - 2z_2 \mathbf{p}_2 \mathbf{y} + z_1^2 \mathbf{p}_1^2 + z_2^2 \mathbf{p}_2^2 \\ \frac{\partial L}{\partial z_1} = -2 \mathbf{p}_1 \mathbf{y} + 2 z_2 \mathbf{p}_1 \mathbf{p}_2 + 2 z_1 \mathbf{p}_1^2 &\Rightarrow \mathbf{p}_1 \mathbf{y} = z_1 \mathbf{p}_1^2 + \end{aligned}$$

$$(9c) \quad \begin{pmatrix} \mathbf{p}_1^T \mathbf{p}_1 & \mathbf{p}_1^T \mathbf{p}_2 \\ \mathbf{p}_2^T \mathbf{p}_1 & \mathbf{p}_2^T \mathbf{p}_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \mathbf{y}^T \mathbf{p}_1 \\ \mathbf{y}^T \mathbf{p}_2 \end{pmatrix}$$

whereas the solution of (9b) is, formally, the regression of  $\mathbf{y}$  on  $\mathbf{p}^\tau$ :

$$(9d) \quad z(t, \tau) = \mathbf{y}(t, \tau)^T \cdot \mathbf{p}^\tau / \mathbf{p}^{\tau T} \cdot \mathbf{p}^\tau$$

### c) The adjoints

As mentioned in the previous Subsection, the adjoint patterns  $\mathbf{q}^\tau$  are a convenient tool to derive the POP coefficients: adjoint patterns are the eigenvectors of the transposed matrix  $\mathcal{B}(\tau)^T$  which has the same eigenvalues as  $\mathcal{B}(\tau)$ . Then

$$\begin{aligned}\mathbf{q}_i^{\tau T} \mathbf{p}_j &= (\lambda_j)^{-1} \mathbf{q}_i^{\tau T} \left( \mathcal{B}(\tau) \mathbf{p}_j^\tau \right) = (\lambda_j)^{-1} \left( \mathcal{B}(\tau)^T \mathbf{q}_i^\tau \right)^T \mathbf{p}_j^\tau \\ &= (\lambda_i / \lambda_j) \mathbf{q}_i^{\tau T} \mathbf{p}_j^\tau\end{aligned}$$

so that  $\mathbf{q}_i^{\tau T} \mathbf{p}_j^\tau = 0$  if  $\lambda_i \neq \lambda_j$ . With proper normalization it is  $\mathbf{q}_j^{\tau T} \mathbf{p}_j^\tau = 1$ . Using this orthonormality, we find

$$\mathbf{y}(t, \tau) = \sum_j z_j(t, \tau) \mathbf{p}_j^\tau \quad \Rightarrow \quad \mathbf{q}_i^{\tau T} \cdot \mathbf{y}(t, \tau) = \sum_j z_j(t, \tau) \mathbf{q}_i^{\tau T} \cdot \mathbf{p}_j^\tau$$

and, after dropping the sub-index  $j$ :

$$(10) \quad z(t, \tau) = \mathbf{q}_j^{\tau T} \cdot \mathbf{y}(t, \tau)$$

i.e., the time coefficient is given by the dot product of the adjoint pattern  $\mathbf{q}_j^\tau$  with the actual field  $\mathbf{y}(t, \tau)$ .

The adjoint patterns  $\mathbf{q}^\tau$  and  $\mathbf{q}^{\tau+1}$  are related to each other in a simple formula, similar to (4):

$$\begin{aligned}\mathcal{B}(\tau)^T \mathbf{q}^\tau &= \lambda \mathbf{q}^\tau \quad \Rightarrow \quad \mathcal{A}(\tau-1)^T \left[ \mathcal{B}(\tau)^T \mathbf{q}^\tau \right] = \lambda \mathcal{A}(\tau-1)^T \mathbf{q}^\tau \\ \Rightarrow \quad \mathcal{B}(\tau-1)^T \left[ \mathcal{A}(\tau+n-1)^T \mathbf{q}^\tau \right] &= \mathcal{B}(\tau-1)^T \left[ \mathcal{A}(\tau-1)^T \mathbf{q}^\tau \right] = \lambda \mathcal{A}(\tau-1)^T \mathbf{q}^\tau\end{aligned}$$

Thus  $\mathcal{A}(\tau-1)^T \mathbf{q}^\tau$  is an adjoint of  $\mathcal{B}(\tau-1)$  if  $\mathbf{q}^\tau$  is an adjoint of  $\mathcal{B}(\tau)$ . Since eigenvectors as well as adjoint are subject to normalization:

$$(11) \quad \mathbf{q}^\tau = s_\tau e^{i\xi_\tau} \mathcal{A}(\tau)^T \mathbf{q}^{\tau+1}$$

Then, using (4) and deleting the sub-index  $\tau$  in  $\xi_\tau$ :

$$\begin{aligned}1 &= \mathbf{q}^\tau \cdot \mathbf{p}^\tau = (s_\tau e^{i\xi_\tau} \mathcal{A}(\tau)^T \mathbf{q}^{\tau+1})^T \cdot \mathbf{p}^\tau = (s_\tau e^{i\xi_\tau}) \mathbf{q}^{\tau+1 T} \cdot \mathcal{A}(\tau) \mathbf{p}^\tau \\ &= (s_\tau / r_\tau e^{i(\xi_\tau - \phi)}) \mathbf{q}^{\tau+1 T} \cdot \mathbf{p}^{\tau+1}\end{aligned}$$

To ensure  $[q^{\tau+1}]^T \cdot p^{\tau+1} = 1$ , one has to set

$$(12) \quad s_\tau = r_\tau \text{ and } \xi_\tau = \phi_\tau.$$

Note that the adjoints are dependent on all eigenvectors, not just the one, or few, that is describing the signal of interest. This is because the adjoint  $q_j^\tau$  has to be orthogonal to all eigenvectors  $p_k^\tau$  with  $k \neq j$ . In some cases, and in particular in synthetic examples, it might be possible that there is only one determined signal in the data set, whereas all other eigenvectors are just reflecting particularities of the considered sample. Then, unfavorably, the adjoint of the one signal has to be orthogonal to all these random eigenvectors.

In the examples presented below, both approaches, the least square fit (9a/b) and the adjoint-formula (10) have been used to determine the POP coefficients. In most cases the results were almost identical.

### 3) ESTIMATING THE SYSTEM MATRICES

If the system matrices  $\mathfrak{A}(\tau)$  are not given by theoretical theory they have to be estimated from the data. To do so, two different approaches are possible. Either, for each seasonal date  $\tau$  the matrix is estimated explicitly by setting

$$(3.1) \quad \mathfrak{A}(\tau) = \mathfrak{C}_{1,\tau} \mathfrak{C}_{0,\tau}^{-1}$$

If the cycle is short (e.g., if the annual cycle is considered in terms of 3-months means ( $n=4$ )), this method is reasonable; if, however, long cycles (e.g., annual cycle in terms of daily values ( $n=365$ )) are considered, this estimate will be quite noisy. In that case it might be better to fit a periodic expression:

$$(3.2) \quad \mathfrak{A}(\tau) = \mathfrak{A}_0 + \sum_j \mathfrak{A}_j \sin(j\omega\tau) + \mathfrak{B}_j \cos(j\omega\tau)$$

with constant matrices  $\mathfrak{A}_j$  and  $\mathfrak{B}_j$  minimizing

$$(3.3) \quad \sum_{\tau} \| \mathfrak{A}_0 + \sum_j \mathfrak{A}_j \sin(j\omega\tau) + \mathfrak{B}_j \cos(j\omega\tau) - \mathfrak{C}_{1,\tau} \mathfrak{C}_{0,\tau}^{-1} \|^2$$

$\mathfrak{C}_{1,\tau}$  and  $\mathfrak{C}_{0,\tau}$  could either be explicitly calculated for each seasonal date  $\tau$  or be fitted to the first few harmonics

$$(4.4) \quad \mathfrak{C}_{1,\tau} = \mathfrak{C}_{01} + \sum_j \mathfrak{C}_{j1} \sin(j\omega\tau) + \mathfrak{D}_{j1} \cos(j\omega\tau)$$

$$(4.5) \quad \mathfrak{C}_{0,\tau} = \mathfrak{C}_{00} + \sum_j \mathfrak{C}_{j0} \sin(j\omega\tau) + \mathfrak{D}_{j0} \cos(j\omega\tau)$$

Note that the estimation of the matrices  $\mathfrak{C}_{0k}$  and  $\mathfrak{D}_{jk}$ ,  $k=0,1$ , does not require to have the all matrices  $\mathfrak{C}_{0,\tau}$  and  $\mathfrak{C}_{1,\tau}$  for all  $\tau$  calculated.

Instead they may be obtained by (4.6)

#### 4) EXAMPLES

A number of synthetic and of real-world examples are considered. The first one is on a damped process with a fixed (standing) spatial pattern and a seasonally varying memory. The second example is a real-world, on the QBO, which is described as a propagating feature, whose spatial appearance is independent of the season, but whose memory does.

The third example, is similar to the first in using a damped process to describe the time-dependency, but a seasonally varying pattern.

##### a) Fixed pattern with seasonally varying memory

This is an synthetic example. A series of 30 years of monthly vector data was generated. Thus, the time series consisted of 360 data points, each of it being a 6-dimensional vector. The vector time series was constructed by

$$(5a.1) \quad y(t,t) = z(t,\tau) \cdot p + s$$

with a time coefficient obeying the first-order autoregressive formula

$$(5a.2) \quad z(t,\tau+1) = \alpha(\tau) \cdot z(t,\tau) + s$$

with a seasonally varying memory

$$(5a.3) \quad \alpha(\tau) = \alpha_0 + \alpha_1 \sin(\tau/n \cdot 2\pi)$$

The process (5a.2) is a damped process and not an oscillatory process. It was used  $\alpha_0 = 0.9$  and  $\alpha_1 = 0.2$ . The seasonal march of the memory, (5a.3) is shown in Fig. 5a.1.

The pattern  $p$  (Fig. 5a.2) and also the noise variances  $\sigma_s$  and  $\sigma_s^2 = 1$  were chosen as being independent of the season. Given formula (5a.2/3) and  $\sigma_s^2 = 1$ , the seasonally varying variance  $\sigma_s^2$  of  $z(\bullet, \tau)$  is obtained by solving the matrix equation

$$(5a.4) \quad \begin{pmatrix} \alpha(1) & 1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha(2) & 1 & 0 & & 0 \\ 0 & 0 & \alpha(3) & 1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & \alpha(n-1) & 1 \\ 1 & 0 & 0 & \cdots & 0 & \alpha(n) \end{pmatrix} \cdot \begin{pmatrix} (\sigma_z^1)^2 \\ (\sigma_z^2)^2 \\ (\sigma_z^3)^2 \\ (\sigma_z^n)^2 \\ (\sigma_z^{n-1})^2 \\ (\sigma_z^n)^2 \end{pmatrix} = \begin{pmatrix} (\sigma_s^1)^2 \\ (\sigma_s^2)^2 \\ (\sigma_s^3)^2 \\ (\sigma_s^n)^2 \\ \vdots \\ (\sigma_s^1)^2 \\ (\sigma_s^2)^2 \end{pmatrix}$$

The resultant annual cycle of the variance,  $(\sigma_z^\tau)^2$ , is shown in Figure 5a.3.

Both, a stationary and a cyclostationary POP analysis were performed with the data, yielding consisting results.

In the stationary POP analysis a standing pattern is identified with an eigenvalue  $\lambda = .9$ . The pattern was very similar to the prescribed pattern (Fig. 5a.2), and  $\lambda$  is identical to the mean memory  $\alpha_0$ . It corresponds to an e-folding time of 9.4 months.

The cyclostationary POP analysis yields also one stationary pattern, with an eigenvalue  $\lambda = 0.16$ , which corresponds to the damping over a whole year. the pattern is, again, quite similar to the prescribed (Fig. 5a.2).The mean month-to-month damping of 0.86 is equivalent to an e-folding time of 6.7 months.

The cyclostationary analysis is quite successful in identifying the annual march of the memory (through the  $r^\tau$ -factors) shown in Fig. 5a.1. The stationary analysis indicates, due to its construction, a fixed memory throughout the year (Fig. 5a.1).

Both techniques are correctly capturing the annual march of the variance  $(\sigma_z^\tau)^2$  of the POP coefficient (Fig 5a.3). Note, however, that this seasonal variable variance in the stationary POP model is excited by a seasonally varying forcing (noise) whereas in the cyclostationary POP model it is, at least partly, due to the variable memory.

### b) The Quasi-Biennial Oscillation

Patterns and coefficients appear in conjugate complex pairs.

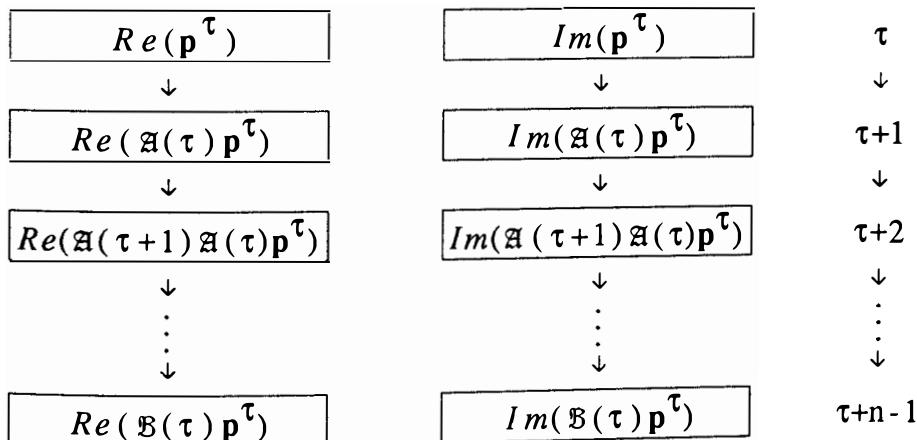
If, at a certain time  $\tau$  we have

$$y(t, \tau) = \gamma p^\tau + (\gamma p^\tau)^* = 2 \operatorname{Re}(\gamma) \cdot \operatorname{Re}(p^\tau) + \operatorname{Im}(\gamma) \cdot \operatorname{Im}(p^\tau)$$

with some complex number  $\gamma$ , then after one time step we have  $y(t, \tau+1) =$

$$y(t, \tau+1) = 2 \operatorname{Re}(\gamma) \cdot \operatorname{Re}(\alpha(\tau)p^\tau) + \operatorname{Im}(\gamma) \cdot \operatorname{Im}(\alpha(\tau)p^\tau)$$

Because of (4),  $\alpha(\tau)p^\tau = \zeta p^{\tau+1}$ . That is  $\alpha(\tau)p^\tau$  is, apart from a complex constant, identical to  $p^{\tau+1}$ . The typical cycle ...



annual cycle of covariance matrices is not smoothed  
POP coefficients are obtained as least square fit

#### STATIONARY POP ANALYSIS

eigenvector no. 1  
eigenvalue: 0.950 0.220      e-folding time: 39.3      period: 27.6

#### CYCLO-STATIONARY POP ANALYSIS

eigenvector no. 1  
eigenvalue: -0.706 0.335      e-folding time: 48.7      period: 27.9

$\tau$	$r_\tau$	real component						$r_\tau$	imaginary component					
		15	20	30	40	50	70		15	20	30	40	50	70
1	94	2	18	34	43	30	17	94	-13	-31	-30	2	36	46
2	99	2	14	32	43	36	24	99	-13	-33	-33	-3	31	43
3	101	-1	9	28	42	38	27	101	-16	-34	-31	-2	28	44
4	101	-3	10	29	42	41	30	101	-17	-32	-21	3	30	45
5	103	-4	15	32	43	36	24	103	-21	-29	-13	10	33	48
6	100	-3	22	36	43	31	21	100	-20	-21	-9	14	40	47
7	99	0	25	39	43	34	20	99	-19	-20	-10	18	41	39
8	96	-1	24	37	45	38	21	96	-17	-21	-10	16	37	41
9	94	-1	23	37	48	40	23	94	-16	-20	-11	11	33	41
10	94	1	24	39	51	40	20	94	-15	-20	-15	3	30	39
11	96	2	25	40	50	33	11	96	-15	-25	-19	1	33	42
12	94	4	24	37	48	28	10	94	-15	-28	-25	-1	34	45

typical evolution beginning in months 1 with real part of POP # 1 (p2 = A(1)p1; p3 = A(2)p2 etc)  
 mo dampg | POP pattern

1	1.000		0.02	0.18	0.34	0.43	0.30	0.17
2	0.945		0.04	0.20	0.36	0.40	0.27	0.13
3	0.938		0.06	0.22	0.37	0.36	0.21	0.05
4	0.949		0.08	0.27	0.34	0.30	0.12	-.04
5	0.966		0.14	0.31	0.30	0.19	-.04	-.22
6	1.002		0.17	0.28	0.24	0.06	-.23	-.33
7	1.010		0.19	0.25	0.18	-.08	-.33	-.34
8	1.000		0.17	0.21	0.09	-.16	-.37	-.41
9	0.965		0.15	0.14	0.02	-.21	-.40	-.43
10	0.911		0.12	0.07	-.04	-.23	-.41	-.40
11	0.862		0.09	0.03	-.09	-.28	-.40	-.34
12	0.831		0.05	-.01	-.12	-.31	-.36	-.30
13	0.782		0.03	-.02	-.14	-.31	-.33	-.27

typical evolution beginning in months 7 with real part of POP # 1 (p2 = A(1)p1; p3 = A(2)p2 etc)  
 mo dampg | POP pattern

7	1.000		0.00	0.25	0.39	0.43	0.34	0.20
9	0.964		0.03	0.27	0.38	0.41	0.29	0.13
10	0.910		0.07	0.29	0.38	0.38	0.19	0.01
11	0.862		0.10	0.32	0.37	0.30	0.03	-.17
12	0.830		0.13	0.31	0.34	0.22	-.09	-.26
13	0.781		0.10	0.28	0.31	0.11	-.17	-.29
2	0.739		0.10	0.26	0.26	0.06	-.18	-.28
3	0.733		0.12	0.24	0.21	-.01	-.22	-.33
4	0.741		0.13	0.20	0.08	-.11	-.30	-.38
5	0.755		0.15	0.13	-.04	-.22	-.35	-.40
6	0.783		0.13	0.00	-.14	-.31	-.39	-.38

7	0.789		0.08	-0.07	-0.21	-0.36	-0.40	-0.30
8	0.782		0.05	-0.11	-0.24	-0.37	-0.37	-0.27

typical evolution beginning in months 1 with imaginary part of POP # 1 (p2 =  
 $A(1)p_1; p_3 = A(2)p_2$  etc)  
mo dampg |                   POP pattern

---

1	1.000		-0.13	-0.31	-0.30	0.02	0.36	0.46
2	0.945		-0.12	-0.28	-0.23	0.07	0.36	0.44
3	0.938		-0.14	-0.25	-0.15	0.16	0.39	0.48
4	0.949		-0.15	-0.18	0.01	0.27	0.47	0.51
5	0.966		-0.16	-0.06	0.16	0.38	0.47	0.47
6	1.002		-0.11	0.11	0.29	0.45	0.45	0.40
7	1.010		-0.04	0.20	0.37	0.46	0.42	0.29
8	1.000		-0.01	0.24	0.37	0.45	0.38	0.21
9	0.965		0.02	0.26	0.37	0.43	0.31	0.13
10	0.911		0.07	0.28	0.38	0.40	0.21	0.01
11	0.862		0.10	0.30	0.37	0.33	0.04	-0.16
12	0.831		0.12	0.31	0.35	0.25	-0.08	-0.25
13	0.782		0.10	0.28	0.33	0.13	-0.15	-0.27

typical evolution beginning in months 7 with imaginary part of POP # 1 (p2 =  
 $A(1)p_1; p_3 = A(2)p_2$  etc)  
mo dampg |                   POP pattern

---

7	1.000		-0.19	-0.20	-0.10	0.18	0.41	0.39
9	0.964		-0.16	-0.13	-0.02	0.23	0.41	0.40
10	0.910		-0.12	-0.07	0.04	0.24	0.41	0.36
11	0.862		-0.09	-0.02	0.10	0.30	0.37	0.31
12	0.830		-0.05	0.02	0.14	0.32	0.34	0.27
13	0.781		-0.03	0.04	0.16	0.32	0.31	0.25
2	0.739		0.00	0.06	0.20	0.31	0.29	0.22
3	0.733		0.00	0.09	0.23	0.31	0.26	0.17
4	0.741		0.01	0.14	0.25	0.29	0.22	0.11
5	0.755		0.05	0.21	0.26	0.24	0.10	-0.02
6	0.783		0.09	0.24	0.25	0.17	-0.04	-0.13
7	0.789		0.12	0.24	0.23	0.07	-0.12	-0.17
8	0.782		0.12	0.22	0.17	0.01	-0.16	-0.24

stationary POP # 1-cycle beginning with real part  
mo dampg |                   POP pattern

---

1	0.975		0.02	0.23	0.38	0.45	0.32	0.16
2	0.950		0.05	0.28	0.40	0.42	0.23	0.05
3	0.927		0.09	0.31	0.41	0.36	0.12	-0.06
4	0.903		0.12	0.33	0.39	0.28	0.01	-0.16
5	0.881		0.14	0.33	0.35	0.19	-0.10	-0.26
6	0.859		0.16	0.32	0.30	0.09	-0.21	-0.35
7	0.837		0.17	0.29	0.23	-0.01	-0.30	-0.41
8	0.816		0.17	0.24	0.15	-0.11	-0.38	-0.46
9	0.795		0.16	0.18	0.06	-0.21	-0.44	-0.48

10	0.776		0.14	0.12	-.03	-.30	-.48	-.48
11	0.756		0.12	0.04	-.12	-.37	-.50	-.45
12	0.737		0.09	-.03	-.21	-.43	-.48	-.40

stationary POP # 1-cycle beginning with imaginary part  
 mo dampg | POP pattern

---

1	0.975		-.17	-.25	-.16	0.11	0.38	0.46
2	0.950		-.16	-.19	-.07	0.21	0.44	0.48
3	0.927		-.14	-.12	0.02	0.29	0.48	0.48
4	0.903		-.12	-.05	0.12	0.37	0.49	0.46
5	0.881		-.09	0.03	0.20	0.42	0.49	0.41
6	0.859		-.06	0.10	0.27	0.46	0.45	0.34
7	0.837		-.02	0.17	0.34	0.46	0.39	0.25
8	0.816		0.02	0.23	0.38	0.45	0.31	0.15
9	0.795		0.06	0.28	0.40	0.41	0.22	0.04
10	0.776		0.09	0.31	0.41	0.35	0.11	-.07
11	0.756		0.12	0.33	0.39	0.28	0.00	-.17
12	0.737		0.14	0.33	0.35	0.19	-.11	-.27

percentage of locally explained variance  
 cyclostationary mode  
 68. 92. 97. 95. 97. 96.  
 stationary mode  
 65. 92. 97. 94. 97. 96.

#### lag-1 forecast skills

	rms	corr-skill
cyclostationary mode	5.0	0.98
stationary mode	5.4	0.98

#### EXAMPLE 3

c:

first seasonal dependent pattern

1	0.245	0.575	0.330	-0.245	-0.575	-0.330
2	0.449	0.539	0.090	-0.449	-0.539	-0.090
3	0.566	0.381	-0.185	-0.566	-0.381	0.185
4	0.546	0.109	-0.436	-0.546	-0.109	0.436
5	0.347	-0.227	-0.573	-0.347	0.226	0.573
6	0.000	-0.500	-0.500	0.000	0.500	0.500
7	-0.347	-0.573	-0.227	0.347	0.573	0.227
8	-0.546	-0.436	0.109	0.546	0.436	-0.109
9	-0.566	-0.185	0.381	0.566	0.185	-0.381
10	-0.449	0.090	0.539	0.449	-0.090	-0.539
11	-0.245	0.330	0.575	0.245	-0.330	-0.575
12	0.000	0.500	0.500	0.000	-0.500	-0.500
0	0.000	0.000	0.000	0.000	0.000	0.000

month damping

month	damping
1	1.000
2	1.073
3	1.100
4	1.073
5	1.000
6	0.900
7	0.800
8	0.727
9	0.700
10	0.727
11	0.800
12	0.900

annual cycle of response and forcing  
variances

month	response	simulated	forcing
1	3.3	2.1	1.0
2	4.3	3.5	1.0
3	5.9	4.4	1.0
4	8.2	5.6	1.0
5	10.4	6.5	1.0
6	11.4	7.2	1.0
7	10.2	6.3	1.0
8	7.6	5.4	1.0
9	5.0	3.9	1.0
10	3.4	2.5	1.0
11	2.8	3.7	1.0
12	2.8	2.6	1.0

1PROGRAM CYCLO: Cyclostationary POP analysis for monthly means and annual cycles

annual cycle of covariance matrices is not smoothed  
POP coefficients are derived from the adjoints

array of local variances  
0.8      0.8      0.8      0.9      0.8      0.8

### STATIONARY POP ANALYSIS

\*\*\*\*\*  
\*\*\*\*\*

eigenvector no. 1  
eigenvalue: 0.716 0.429    e-folding time: 5.5    period: 11.6  
Ei R: 0.25 0.41 0.16 -0.25 -0.40 -0.16  
Ei I: -0.33 0.03 0.37 0.34 -0.03 -0.37  
Ad R: 0.25 0.29 0.21 -0.29 -0.45 -0.22  
Ad I: 0.66 0.09 -0.25 -0.04 0.24 0.47

### CYCLO-STATIONARY POP ANALYSIS

\*\*\*\*\*

eigenvector no. 1

eigenvalue: 0.175 0.000 e-folding time: 6.9 period: \*\*\*\*\*  
 Ei R: 0.26 0.56 0.38 -0.20 -0.58 -0.31  
 Ad R: 1.77 -0.29 1.78 -0.64 0.69 -0.98

cyclostationary POP # 1 real part  
 mo r(mo) | POP pattern

1	1.005		0.26	0.56	0.38	-.20	-.58	-.31
2	0.942		0.44	0.53	0.08	-.48	-.53	-.05
3	1.005		0.56	0.41	-.17	-.57	-.37	0.20
4	1.035		0.53	0.12	-.43	-.56	-.08	0.45
5	0.998		0.36	-.23	-.54	-.33	0.21	0.60
6	0.841		0.01	-.51	-.50	-.02	0.48	0.50
7	0.826		-.33	-.58	-.26	0.30	0.58	0.22
8	0.822		-.54	-.44	0.12	0.55	0.44	-.09
9	0.672		-.56	-.21	0.43	0.55	0.18	-.35
10	1.082		-.42	0.02	0.58	0.47	-.13	-.50
11	0.584		-.27	0.33	0.59	0.23	-.29	-.58
12	0.736		0.03	0.52	0.48	-.02	-.47	-.53
1	1.005		0.26	0.56	0.38	-.20	-.58	-.31

typical evolution beginning in months 1 with real part of POP # 1 (p2 =  
 A(1)p1; p3 = A(2)p2 etc)  
 mo dampg | POP pattern

1	1.000		0.26	0.56	0.38	-.20	-.58	-.31
2	1.005		0.44	0.54	0.08	-.49	-.53	-.05
3	0.947		0.53	0.39	-.16	-.54	-.35	0.19
4	0.951		0.50	0.11	-.41	-.53	-.08	0.43
5	0.985		0.35	-.23	-.54	-.33	0.21	0.59
6	0.982		0.01	-.50	-.49	-.02	0.47	0.50
7	0.826		-.27	-.48	-.22	0.25	0.48	0.18
8	0.683		-.37	-.30	0.08	0.38	0.30	-.06
9	0.561		-.31	-.12	0.24	0.31	0.10	-.19
10	0.377		-.16	0.01	0.22	0.18	-.05	-.19
11	0.408		-.11	0.13	0.24	0.09	-.12	-.24
12	0.238		0.01	0.12	0.11	0.00	-.11	-.13
13	0.175		0.05	0.10	0.07	-.03	-.10	-.05

typical evolution beginning in months 7 with real part of POP # 1 (p2 =  
 A(1)p1; p3 = A(2)p2 etc)  
 mo dampg | POP pattern

7	1.000		-.33	-.58	-.26	0.30	0.58	0.22
9	0.822		****	-.57	1.27	1.18	0.37	-.45
10	0.553		-.40	-.08	0.92	0.68	-.36	-.50
11	0.598		-.34	0.47	0.80	0.28	-.40	-.76

12	0.349		0.02	0.41	0.40	-.01	-.38	-.43
13	0.257		0.15	0.32	0.22	-.10	-.34	-.19
2	0.258		0.25	0.31	0.05	-.28	-.31	-.03
3	0.243		0.31	0.22	-.09	-.31	-.20	0.11
4	0.244		0.29	0.07	-.24	-.31	-.05	0.25
5	0.253		0.21	-.13	-.31	-.19	0.12	0.34
6	0.252		0.00	-.29	-.29	-.01	0.27	0.29
7	0.212		-.16	-.28	-.12	0.15	0.28	0.11
8	0.175		-.21	-.17	0.05	0.22	0.17	-.04

stationary POP # 1-cycle beginning with real part  
 mo dampg | POP pattern

1	0.835		0.25	0.41	0.16	-.25	-.40	-.16
2	0.696		0.38	0.34	-.05	-.38	-.33	0.06
3	0.581		0.41	0.17	-.25	-.41	-.16	0.25
4	0.485		0.32	-.05	-.38	-.32	0.05	0.38
5	0.405		0.14	-.25	-.39	-.14	0.25	0.39
6	0.338		-.08	-.39	-.30	0.08	0.38	0.30
7	0.282		-.28	-.41	-.12	0.28	0.40	0.12
8	0.235		-.40	-.32	0.09	0.40	0.30	-.09
9	0.196		-.40	-.14	0.28	0.40	0.12	-.28
10	0.164		-.29	0.09	0.39	0.30	-.09	-.39
11	0.137		-.10	0.28	0.38	0.10	-.28	-.38
12	0.114		0.12	0.40	0.27	-.12	-.39	-.27

stationary POP # 1-cycle beginning with imaginary part  
 mo dampg | POP pattern

1	0.835		-.33	0.03	0.37	0.34	-.03	-.37
2	0.696		-.16	0.23	0.40	0.16	-.23	-.40
3	0.581		0.06	0.38	0.31	-.06	-.37	-.31
4	0.485		0.26	0.41	0.14	-.26	-.40	-.14
5	0.405		0.39	0.33	-.07	-.39	-.32	0.08
6	0.338		0.40	0.15	-.27	-.41	-.14	0.27
7	0.282		0.30	-.07	-.38	-.31	0.07	0.38
8	0.235		0.12	-.27	-.39	-.12	0.26	0.39
9	0.196		-.10	-.39	-.29	0.10	0.38	0.29
10	0.164		-.29	-.41	-.10	0.29	0.39	0.10
11	0.137		-.40	-.30	0.11	0.40	0.29	-.11
12	0.114		-.39	-.12	0.29	0.40	0.11	-.30

statistics of cyclostationary mode and of stationary mode  
 real parts

month	varaince	cyclo.	stat.	correlation
1		2.27*****		0.000
2		4.11*****		0.000
3		4.51*****		0.000
4		5.20*****		0.000
5		6.55*****		0.000
6		7.26*****		0.000
7		6.26*****		0.000
8		4.97*****		0.000

9	4.18*****	0.000
10	2.76*****	0.000
11	4.29*****	0.000
12	2.59*****	0.000

percentage of locally explained variance

cyclostationary mode

84. 87. 85. 82. 87. 84.

lag-1 forecast skills

cyclostationary mode	rms	corr-skill
	1.1	0.87

#### 4.) PREDICTIONS

Using Kalman filter (Honerkamp and Weese, 1988?)

#### 6. TECHNICAL ASPECTS

## REFERENCES

Blumenthal, B., 1990: *Predictability of a coupled ocean-atmosphere model.* submitted to J. Climate

Hasselmann, K., 1988: PIPs and POPs: *the reduction of complex dynamical systems using principal interaction and oscillation patterns.* J. Geophys. Res. ??, 11015-11021

Honerkamp, J. and U. Weese, 1989: *State state models, PIPs, POPs and EOFs.* Internal report THEP 89/1 of University of Freiburg

Ortiz et al., 1990: ...., submitted to J. Climate

Storch, H.v.; T. Bruns, I. Fischer-Bruns and K. H. Hasselmann, 1988: ... J. Geophys Res.

## Figure Captions

### Figure 5a.1

The seasonally march of the memory in Example 5a.1, as prescribed in the synthetically generated vector time series, as reconstructed by the stationary POP analysis (a constant) and by the cyclostationary POP analysis.

### Figure 5a.2

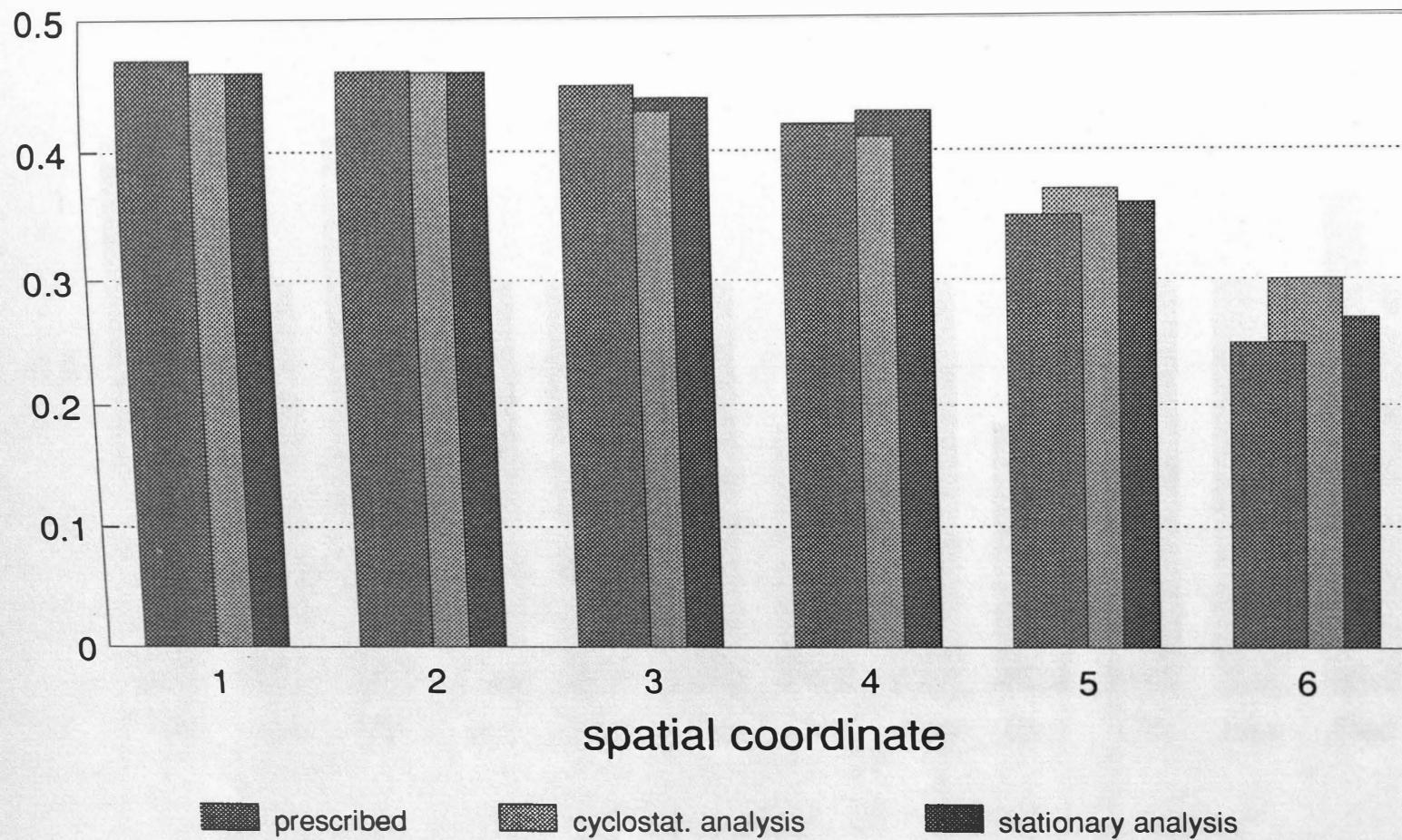
The standing, and seasonally fixed, pattern used in Example 5a.1: as prescribed in the synthetically generated vector time series, and as identified in the stationary POP analysis and in the cyclostationary POP analysis. The number of spatial coordinates is arbitrarily set to 6.

### Figure 5a.3

The seasonally march of the variance  $(\sigma_z^{\tau})^2$  of the (real) POP coefficient  $z$  in Example 5a.1, as prescribed in the synthetically generated vector time series, as reconstructed by the stationary POP analysis and by the cyclostationary POP analysis.

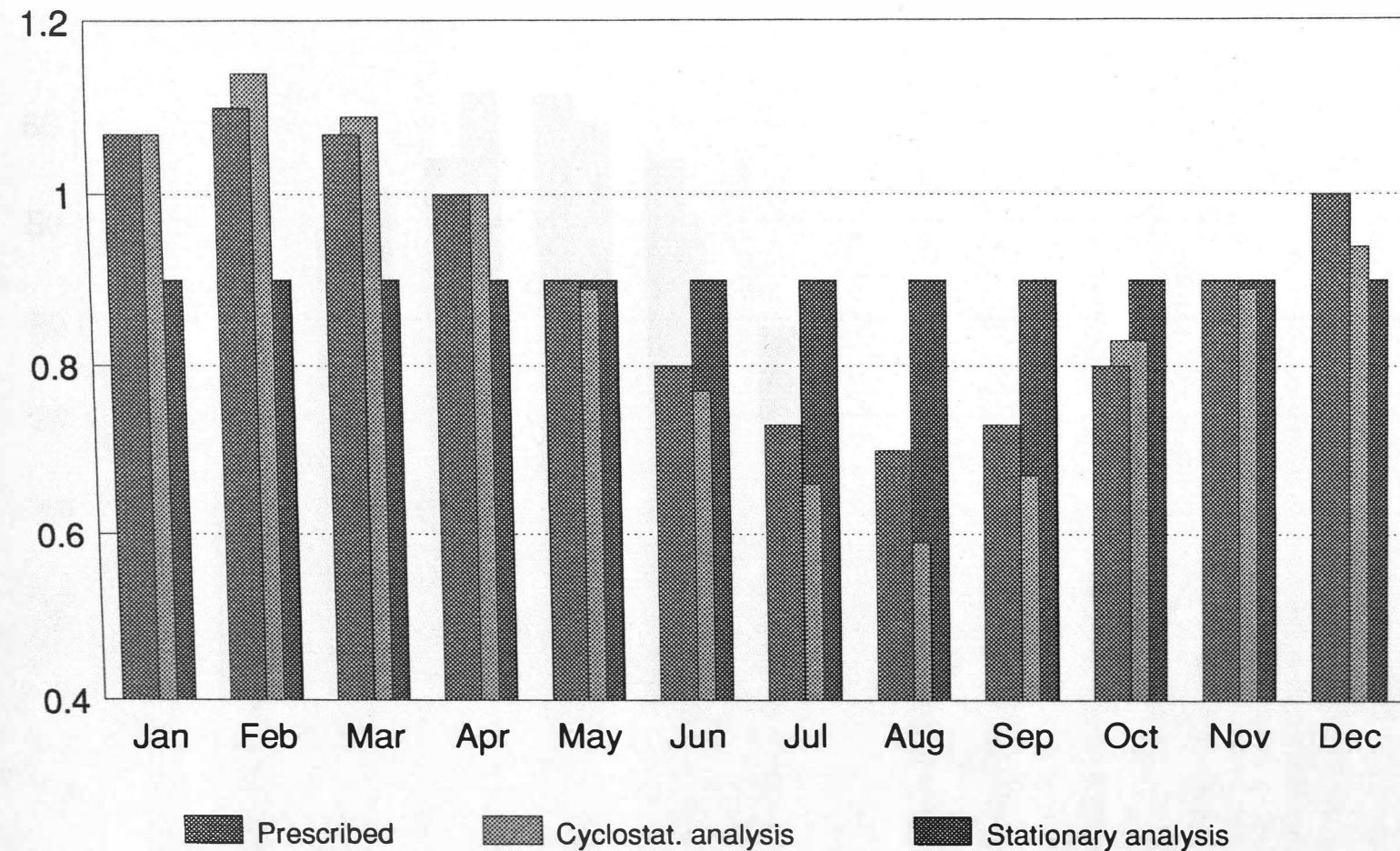
Constant Pattern with Varying Memory

# Patterns



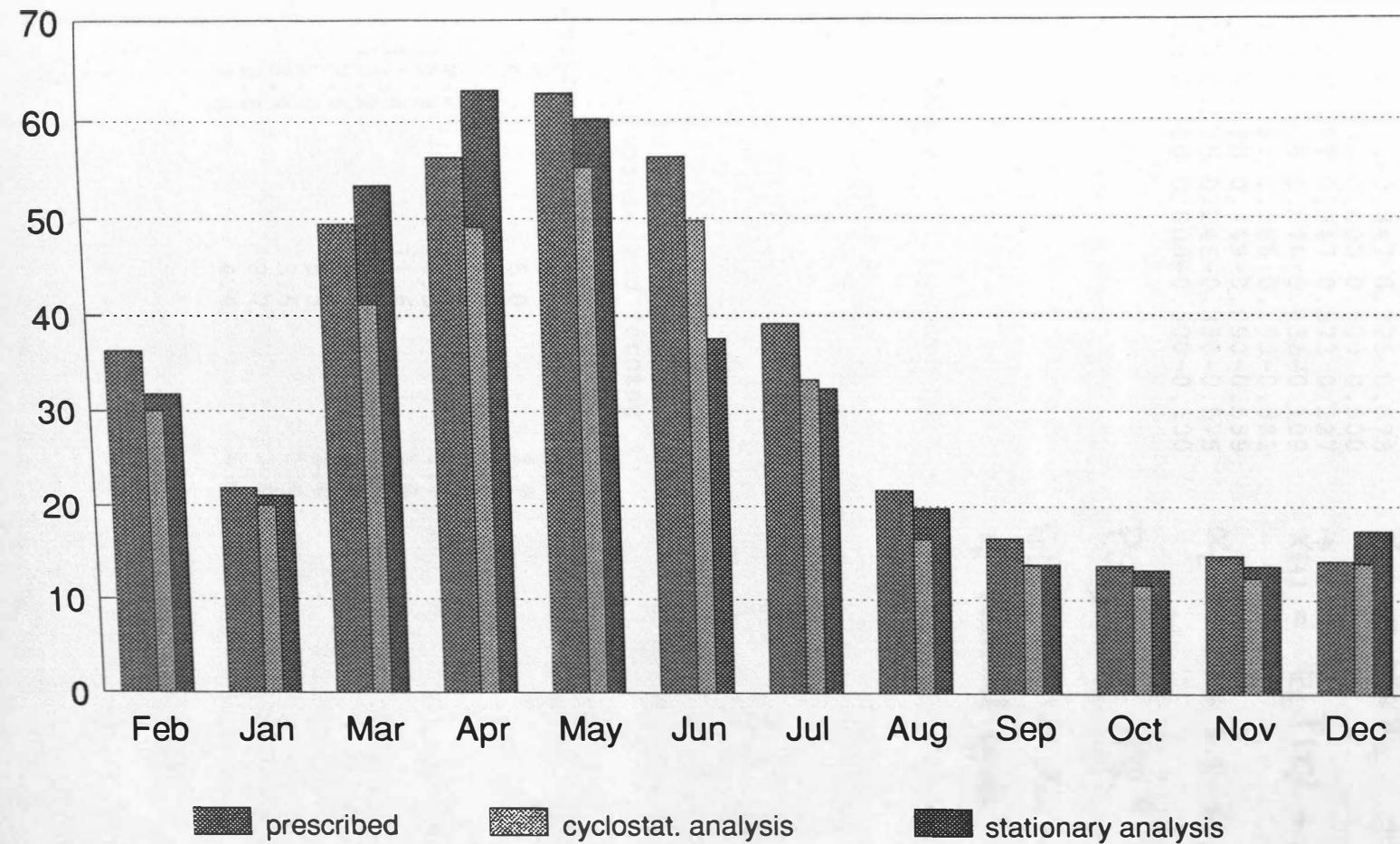
cyclo1

Fixed Pattern with Varying Memory  
Damping Rates



Fixed Pattern with Varying Memory

# Annual March of the Variances



cyclo3

$\tau$	$P(\tau)$					
first seasonal dependent pattern						
1	0.245	0.575	0.330	-0.245	-0.575	-0.330
2	0.449	0.539	0.090	-0.449	-0.539	-0.090
3	0.566	0.381	-0.185	-0.566	-0.381	0.185
4	0.546	0.109	-0.436	-0.546	-0.109	0.436
5	0.347	-0.227	-0.573	-0.347	0.226	0.573
6	0.000	-0.500	-0.500	0.000	0.500	0.500
7	-0.347	-0.573	-0.227	0.347	0.573	0.227
8	-0.546	-0.436	0.109	0.546	0.436	-0.109
9	-0.566	-0.185	0.381	0.566	0.185	-0.381
10	-0.449	0.090	0.539	0.449	-0.090	-0.539
11	-0.245	0.330	0.575	0.245	-0.330	-0.575
12	0.000	0.500	0.500	0.000	-0.500	-0.500

month damping

month	damping
1	1.000
2	1.073
3	1.100
4	1.073
5	1.000
6	0.900
7	0.800
8	0.727
9	0.700
10	0.727
11	0.800
12	0.900

$\alpha_\tau$

## Synthesical Example

$$z_{t+1} = \alpha_t z_t + \text{noise}$$

$$\rightarrow X(t) = z_t \cdot P(\tau) + \text{noise}$$

$$\alpha_t = 0.9 + 0.2 \cdot \sin\left(\frac{\pi}{12} \cdot t\right)$$

$$P(\tau), X(t) \in \mathbb{R}^6$$

noise independent  
of "annual cycle".

## annual cycle of response and forcing

variances

month	response	simulated	forcing
1	3.3	3.8	1.0
2	4.3	5.0	1.0
3	5.9	7.4	1.0
4	8.2	8.9	1.0
5	10.4	10.5	1.0
6	11.4	11.6	1.0
7	10.2	10.2	1.0
8	7.6	8.3	1.0
9	5.0	5.5	1.0
10	3.4	3.6	1.0
11	2.8	3.7	1.0
12	2.8	4.4	1.0

CYCLO-STATIONARY POP ANALYSIS

eigenvector no. 1  
eigenvalue: 0.323 0.000 e-folding time: 10.6 period: \*\*\*\*\*

cyclostationary POP # 1 real part  
mo r(mo) | POP pattern

1	1.000		0.28	0.62	0.29	-.24	-.55	-.32
2	1.246		0.48	0.53	0.12	-.39	-.56	-.09
3	1.009		0.56	0.37	-.17	-.57	-.40	0.21
4	1.051		0.55	0.12	-.45	-.55	-.11	0.41
5	0.988		0.37	-.20	-.58	-.38	0.20	0.55
6	0.853		0.03	-.47	-.53	-.03	0.48	0.52
7	0.873		-.32	-.58	-.26	0.35	0.57	0.22
8	0.746		-.53	-.43	0.09	0.56	0.45	-.13
9	0.765		-.53	-.19	0.39	0.61	0.17	-.36
10	0.738		-.50	0.03	0.56	0.43	-.05	-.51
11	0.954		-.24	0.36	0.58	0.19	-.37	-.55
12	0.827		0.01	0.50	0.53	0.01	-.48	-.49
1	1.000		0.28	0.62	0.29	-.24	-.55	-.32

month	variance	cyclo.
1		3.86
2		4.37
3		7.99
4		9.48
5		11.58
6		12.87
7		11.00
8		8.95
9		6.21
10		4.48
11		3.57

## STATIONAR Y ANALYSIS

eigenvalue: 0.767 0.463 e-folding time: 9.1 period: 11.6  
Ei R: -0.04 0.32 0.38 0.04 -0.32 -0.35  
Ei I: -0.41 -0.26 0.16 0.42 0.25 -0.17

stationary POP # 1-cycle beginning with real part  
mo dampg | POP pattern

---

1	0.896		-.04	0.32	0.38	0.04	-.32	-.35
2	0.803		0.18	0.40	0.24	-.18	-.40	-.22
3	0.720		0.35	0.37	0.03	-.35	-.37	-.02
4	0.646		0.41	0.24	-.18	-.42	-.23	0.19
5	0.579		0.36	0.03	-.35	-.37	-.03	0.33
6	0.519		0.21	-.18	-.41	-.21	0.19	0.39
7	0.465		-.01	-.34	-.35	0.01	0.35	0.33
8	0.417		-.22	-.41	-.20	0.23	0.41	0.18
9	0.374		-.37	-.35	0.02	0.38	0.35	-.03
10	0.335		-.41	-.20	0.22	0.42	0.19	-.22
11	0.300		-.34	0.01	0.37	0.34	-.02	-.36
12	0.269		-.17	0.22	0.41	0.17	-.23	-.39

stationary POP # 1-cycle beginning with imaginary part  
mo dampg | POP pattern

---

1	0.896		-.41	-.26	0.16	0.42	0.25	-.17
2	0.803		-.38	-.06	0.33	0.38	0.05	-.32
3	0.720		-.23	0.16	0.41	0.23	-.17	-.39
4	0.646		-.02	0.33	0.37	0.02	-.33	-.34
5	0.579		0.20	0.41	0.22	-.20	-.41	-.20
6	0.519		0.36	0.36	0.01	-.37	-.36	0.00
7	0.465		0.42	0.22	-.20	-.42	-.21	0.20
8	0.417		0.35	0.01	-.36	-.36	0.00	0.35
9	0.374		0.19	-.20	-.41	-.19	0.21	0.39
10	0.335		-.03	-.36	-.34	0.03	0.36	0.32
11	0.300		-.24	-.41	-.18	0.25	0.41	0.16
12	0.269		-.38	-.34	0.04	0.39	0.34	-.05

## Cyclostationary POP analysis

Example 5b:

# The Quasi-Biennial Oscillation in the Equatorial Stratosphere

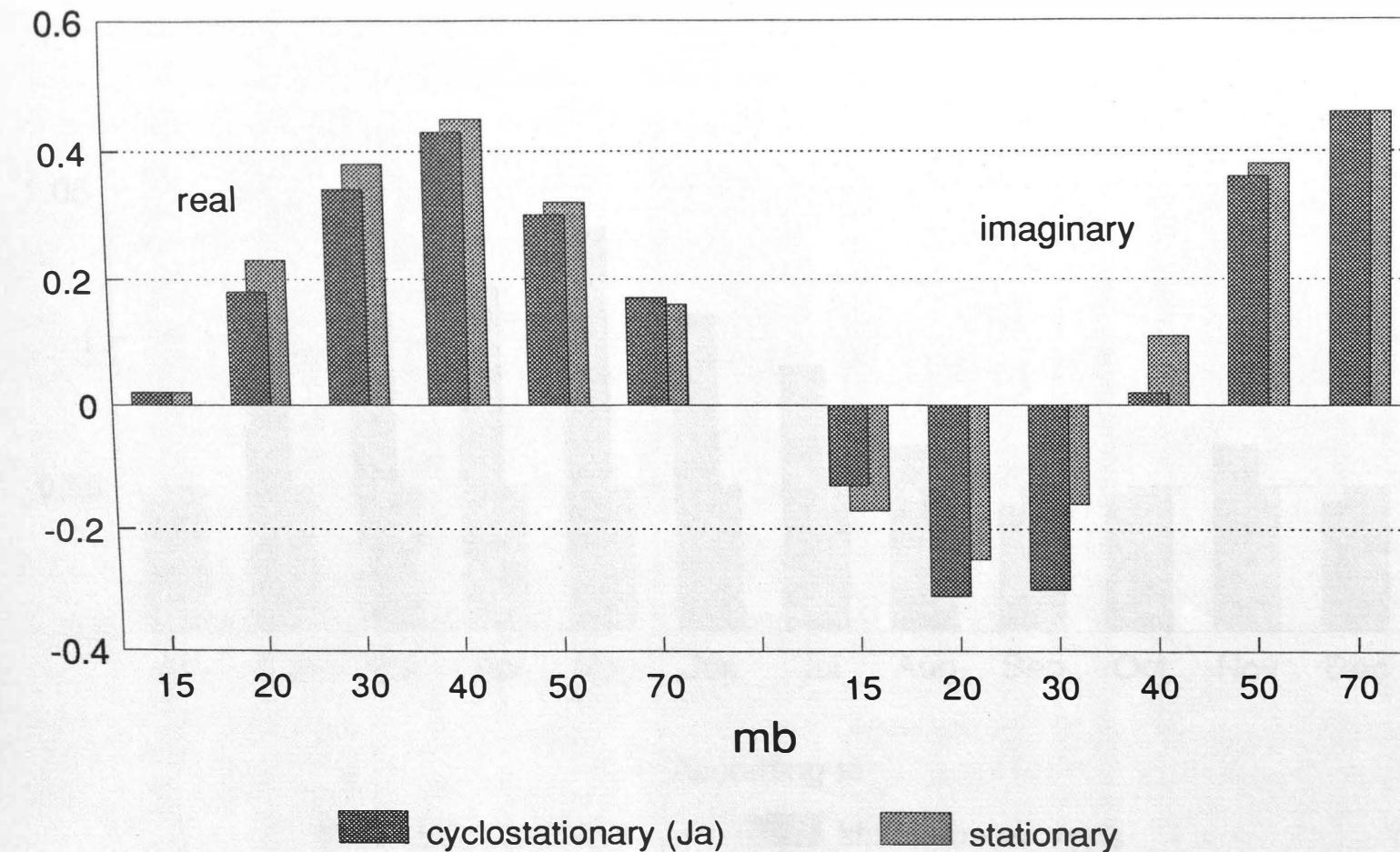
text20

The QBO

# The Data

- Zonally Averaged Zonal Wind in the Equatorial Stratosphere
- at 15, 20, 30, 40, 50 and 70 mb
- Anomalies of monthly means  
from January 1953 through August 1989
- supplied by Meteorologisches Institut  
Freie Universitaet Berlin
- see also: Xu, J., 1990: On the relationship  
between the SO and the QBO. MPI Report #48

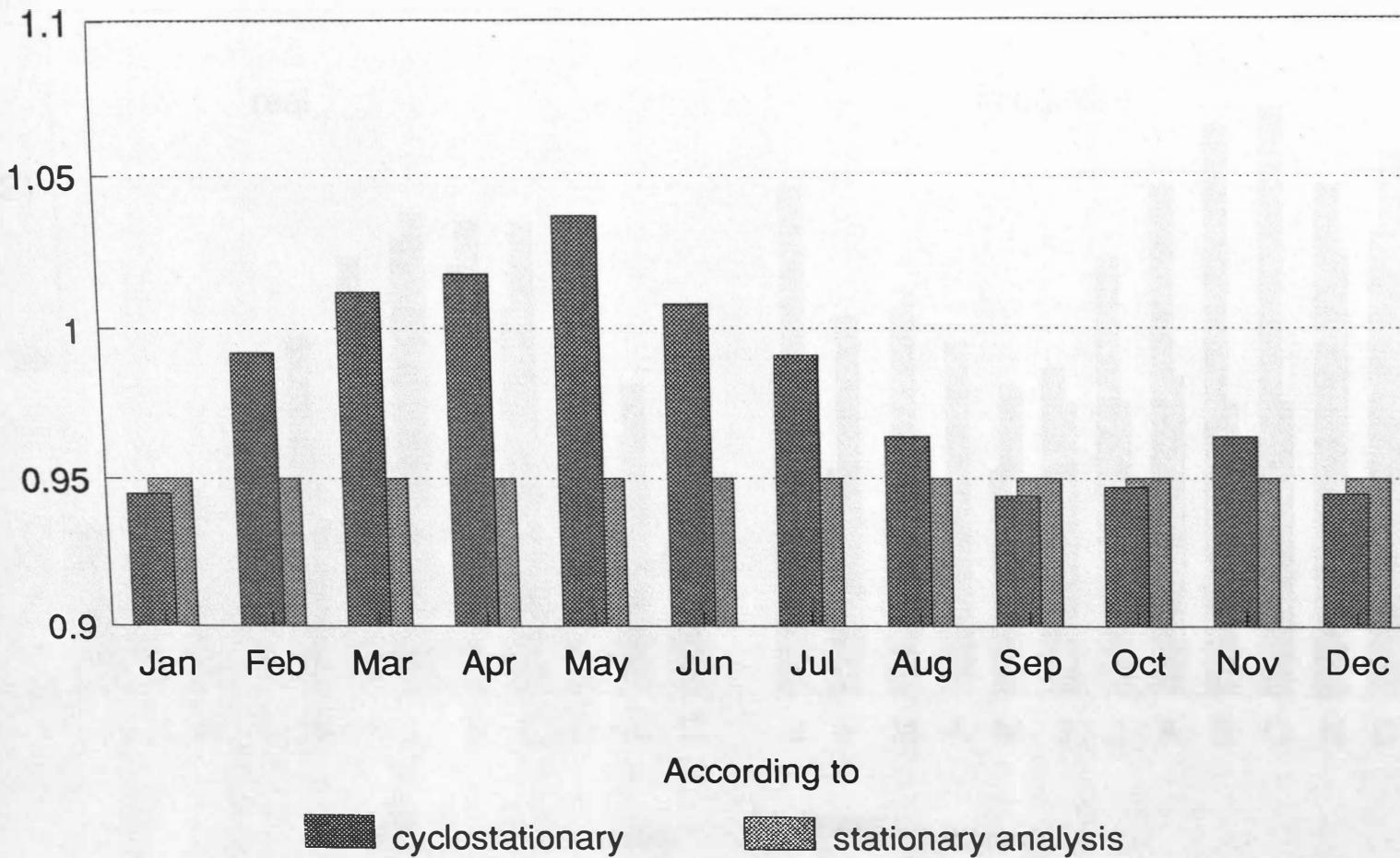
# Principal Oscillation Patterns



cyclb1

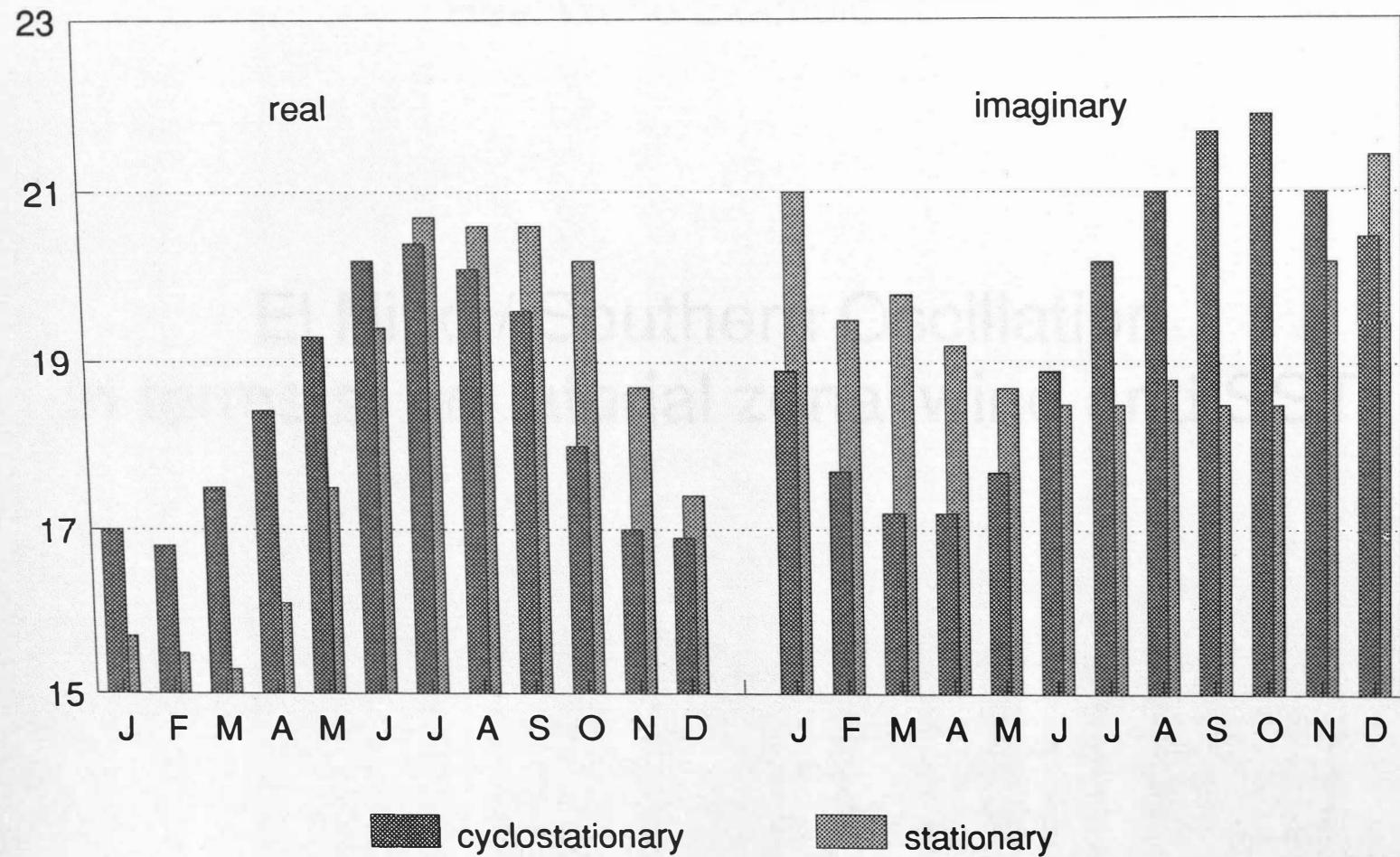
# The QBO

## Seasonal Cycle of Damping Rates



The QBO

# Seasonal Cycle of Variance



cyclob3

## Cyclostationary POP analysis

Real World Example 5d:

El Nino / Southern Oscillation  
in terms of equatorial zonal wind and SST

# ENSO

## The Data

- Monthly anomalies of
  - zonal surface wind
  - SSTalong the Equator from the African coast to Peruvian coast on a  $5^{\circ}$  deg grid.
- January 1951 through December 1986 (36 years of monthly data)
- Data supplied by Tim Barnett (SIO)
- see also Storch,Weese,Xu: Z. Meteor. 40, 1990

# ENSO

## Data Preprocessing

- data have been (Fourier) filtered so that
  - variability on time scales < 6 months is suppressed
  - variability on time scales > 12 months is unchanged
  - in between: cosine tapering
- data have been normalized so that the 24-point vector of SST anomalies and the 24-point vector of the zonal wind anomalies have the same spatially averaged variance 1.
- The 48-dimensional vector, combined from the filtered and normalized SST and zonal wind vectors, is EOF analysed. The first 10 EOFs, explaining about 75% of the (filtered and normalized) variance are retained.

# ENSO Results

- A stationary and a cyclostationary POP analysis is done.
  - Modes with a memory > 0.8 the averaged local memory are retained.
  - TWO modes are found
  - mode period e-folding time
- 

1	30	25	stationary
1	29	18	cyclostationary
2	46	14	stationary
2	40	13	cyclostationary

- In the following only mode 1 is considered.

# ENSO Damping Rates

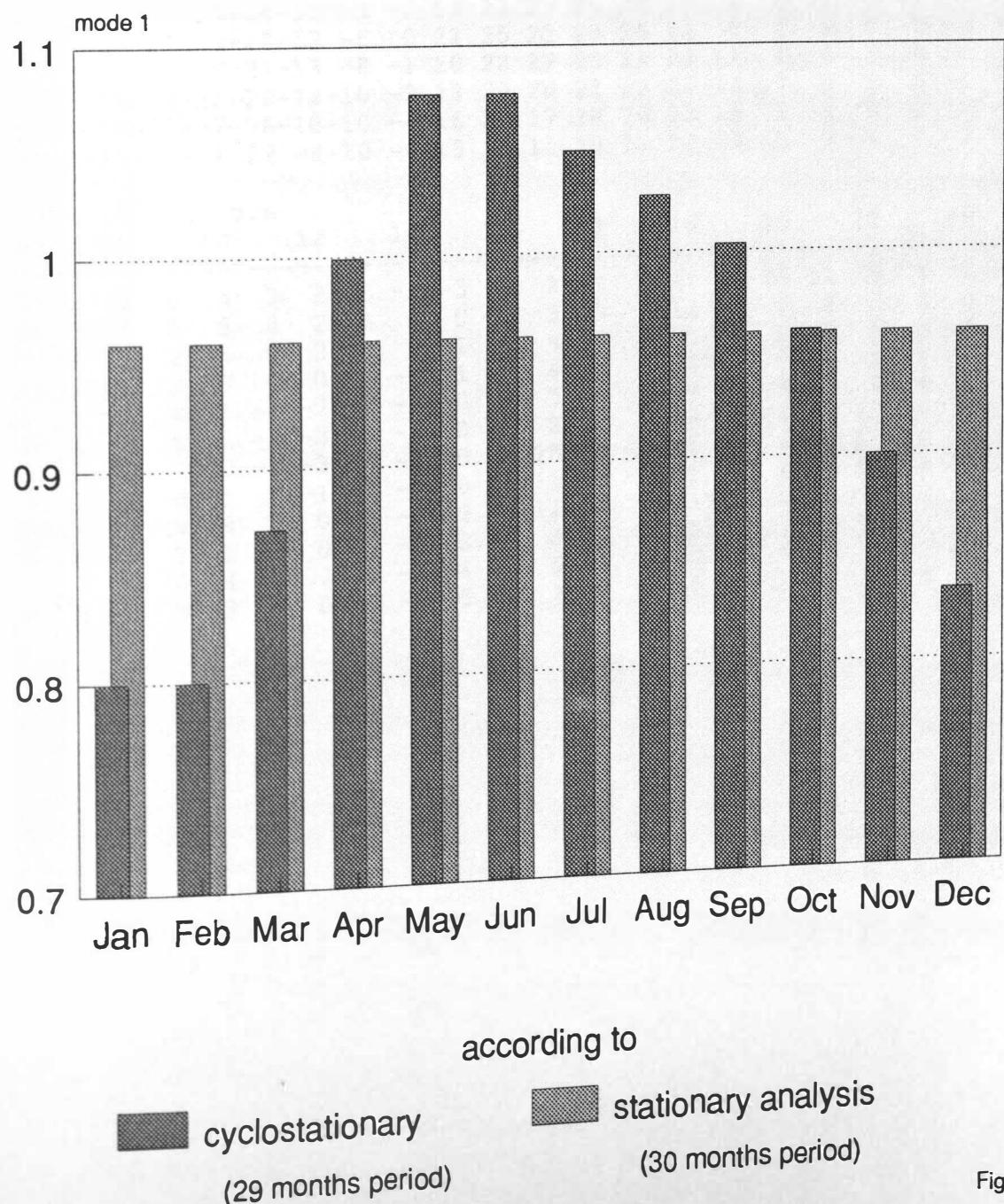


Fig5d

ENSO: CYCLOSTATIONARY ANALYSIS  
\*\*\*\*\*

imaginary component

zonal surface wind

	6	8	10	12	14	16	18	16	14	12	10	8												
1	3	-3	-14	-19	-23	-18	-5	-11	-3	16	10	14	12	14	20	15	6	1	-5	-1	-4	-2	-3	-3
2	2	-4	-12	-17	-20	-16	-3	-11	-2	17	9	13	10	12	17	12	3	0	-7	-3	-5	-2	-3	-2
3	2	-3	-10	-15	-18	-15	-3	-10	0	16	10	13	10	11	13	7	0	-2	-7	-5	-6	-3	-3	-2
4	2	-2	-8	-12	-17	-15	-5	-7	1	16	16	16	14	12	11	3	0	-3	-4	-5	-6	-3	-3	-1
5	2	-1	-7	-11	-18	-16	-8	-3	3	19	23	19	19	17	12	1	0	-2	0	-3	-5	-3	-3	0
6	3	-1	-9	-13	-22	-19	-10	-2	3	22	26	22	23	21	14	1	0	-1	3	-1	-5	-4	-3	0
7	4	-1	-11	-16	-27	-24	-12	-3	2	23	27	23	25	24	18	3	0	-1	5	-1	-5	-4	-4	0
8	5	-2	-13	-21	-33	-28	-13	-5	0	21	25	23	25	25	21	7	1	-1	5	0	-5	-4	-4	-1
9	6	-3	-15	-24	-36	-31	-13	-8	-1	20	22	22	23	24	23	11	4	0	3	0	-5	-3	-4	-2
10	6	-3	-16	-24	-35	-29	-12	-10	-3	18	20	20	21	22	24	14	7	2	2	0	-4	-2	-3	-3
11	5	-3	-16	-23	-32	-26	-10	-10	-4	16	17	17	18	20	23	16	8	3	0	1	-3	-1	-3	-3
12	4	-3	-15	-21	-28	-22	-8	-10	-4	15	15	15	15	17	22	17	8	3	-1	0	-3	-1	-2	-3

sea surface temperature

	6	8	10	12	14	16	18	16	14	12	10	8												
1	10	8	2	3	4	0	2	3	-1	-3	-4	2	-4	6	11	12	10	11	9	7	4	2	4	6
2	10	7	1	3	5	0	2	4	0	0	-2	3	-3	7	11	11	8	8	7	3	0	-2	0	2
3	7	4	1	2	5	0	2	4	0	1	0	4	0	7	11	10	6	6	5	0	-2	-3	-2	0
4	4	2	1	2	3	0	0	1	-1	1	0	4	2	7	10	9	5	6	5	1	0	0	0	0
5	2	1	2	4	1	-1	-2	-2	-4	0	2	5	5	8	9	10	5	7	7	4	4	5	4	1
6	2	1	3	5	2	-2	-3	-4	-6	0	3	8	5	8	9	12	6	10	10	6	7	8	6	1
7	3	2	5	7	3	-2	-3	-4	-7	0	4	10	5	9	11	14	8	11	12	8	9	10	7	1
8	5	3	7	9	5	-1	-1	-2	-6	0	5	11	5	8	12	15	9	13	14	9	10	10	7	2
9	6	4	7	9	6	0	0	-1	-5	0	3	11	4	6	12	15	10	13	15	10	10	10	9	5
10	8	6	7	8	5	0	0	0	-4	-2	0	8	2	4	12	15	11	14	15	10	11	10	11	9
11	8	7	6	6	4	0	0	0	-4	-4	-3	5	0	3	10	14	11	13	14	11	11	9	11	11
12	9	8	3	5	3	0	0	1	-4	-5	-5	2	-2	3	10	13	10	12	12	9	9	7	9	10

# ENSO CYCLOSTATIONARY ANALYSIS

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real component

zonal surface wind

	6	8	10	12	14	16	18	16	14	12	10	8	
1	4	0	-6-10-15-15	-4	-6-10-17	8	1	4	6	15	20	9	5
2	3	0	-5-8-11-12	-1	-8-14-25	1	-3	0	3	16	25	10	11
3	2	0	-3-5-5-7	2	-8-16-37	-8-11	-8	-1	14	26	9	11	4
4	2	0	0	1	-1	6	-6-16-49-16-19-14	-8	6	21	6	8	3
5	1	0	3	3	7	2	7	-3-13-51-15-21-16-12	-2	12	3	6	1
6	0	1	5	6	9	4	6	0-9-44	-8-17-14-13	-8	5	0	5
7	1	2	5	7	8	3	4	3-6-36	-1-12-10-10	-8	1	0	4
8	1	2	3	4	3	0	1	4-5-28	3-7	-5	-5	1	-1
9	2	1	0	0	-3	-6	-1	2-4-21	7	-3	0	0	2
10	3	0	-2	-4-10-12	-3	0	-5-16	10	0	3	4	4	5
11	4	0	-4	-8-15-16	-5	-2	-6-12	12	3	5	6	8	9
12	4	0	-5-10-17-17	-6	-4	-7-12	12	4	6	7	12	14	6

sea surface temperature

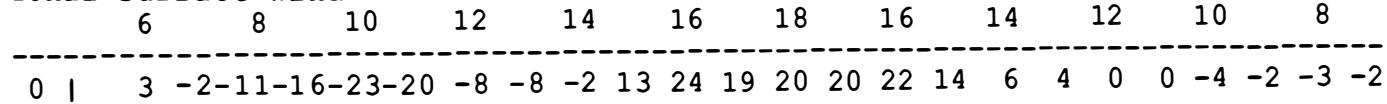
	6	8	10	12	14	16	18	16	14	12	10	8	
1	7	7	14	8	11	11	18	10	0-9-12	-3	-2	-7	6
2	9	8	14	9	13	13	22	14	0-9-14	-6	-5	-9	6
3	8	7	14	8	15	15	26	16	2-8-14	-8	-7-12	4	9
4	5	4	13	5	14	15	27	17	6-6-12-10	-6-14	0	2	9
5	1	2	12	2	9	13	22	14	7-4-10-11	-2-15	-4	-3	2
6	-1	0	10	0	4	11	16	10	5-4	-9-12	0-15	-7	-6
7	-3	0	9	0	1	8	11	6	2-5	-8-10	1-14	-7	-6
8	-3	0	9	2	1	7	9	3	0-5	-6	-7	2-11	-6
9	-2	0	10	4	3	6	8	3	-1	-4	-4	-3	0
10	0	0	11	6	5	6	9	3	-1	-3	-3	0	2
11	2	2	13	7	7	7	11	5	-1	-4	-5	0	2
12	4	5	14	8	8	9	13	7	-1	-6	-8	-1	1

ENSO: STATIONARY POP ANALYSIS

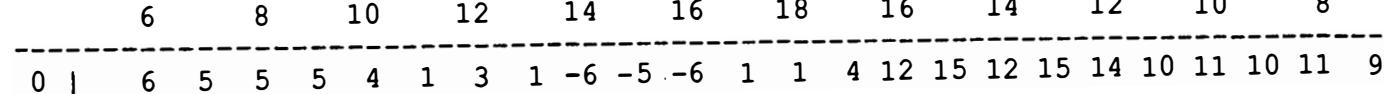
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real component

zonal surface wind

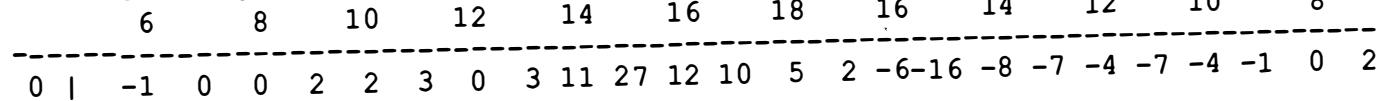


sea surface temperature

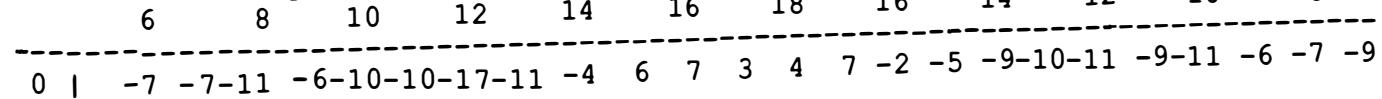


imaginary component

zonal surface wind



sea surface temperature



ENSO

# Seasonal Cycle of Variance

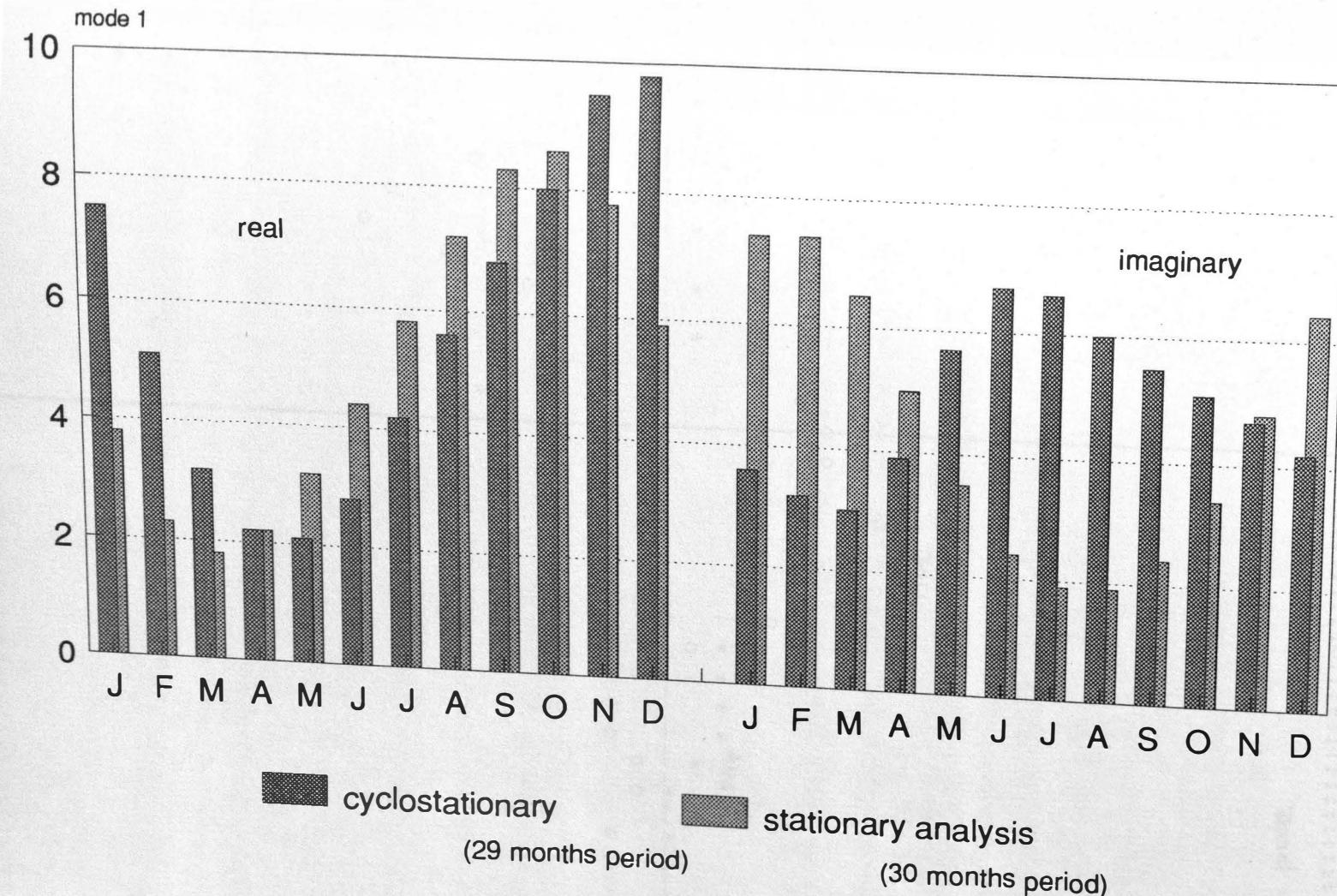
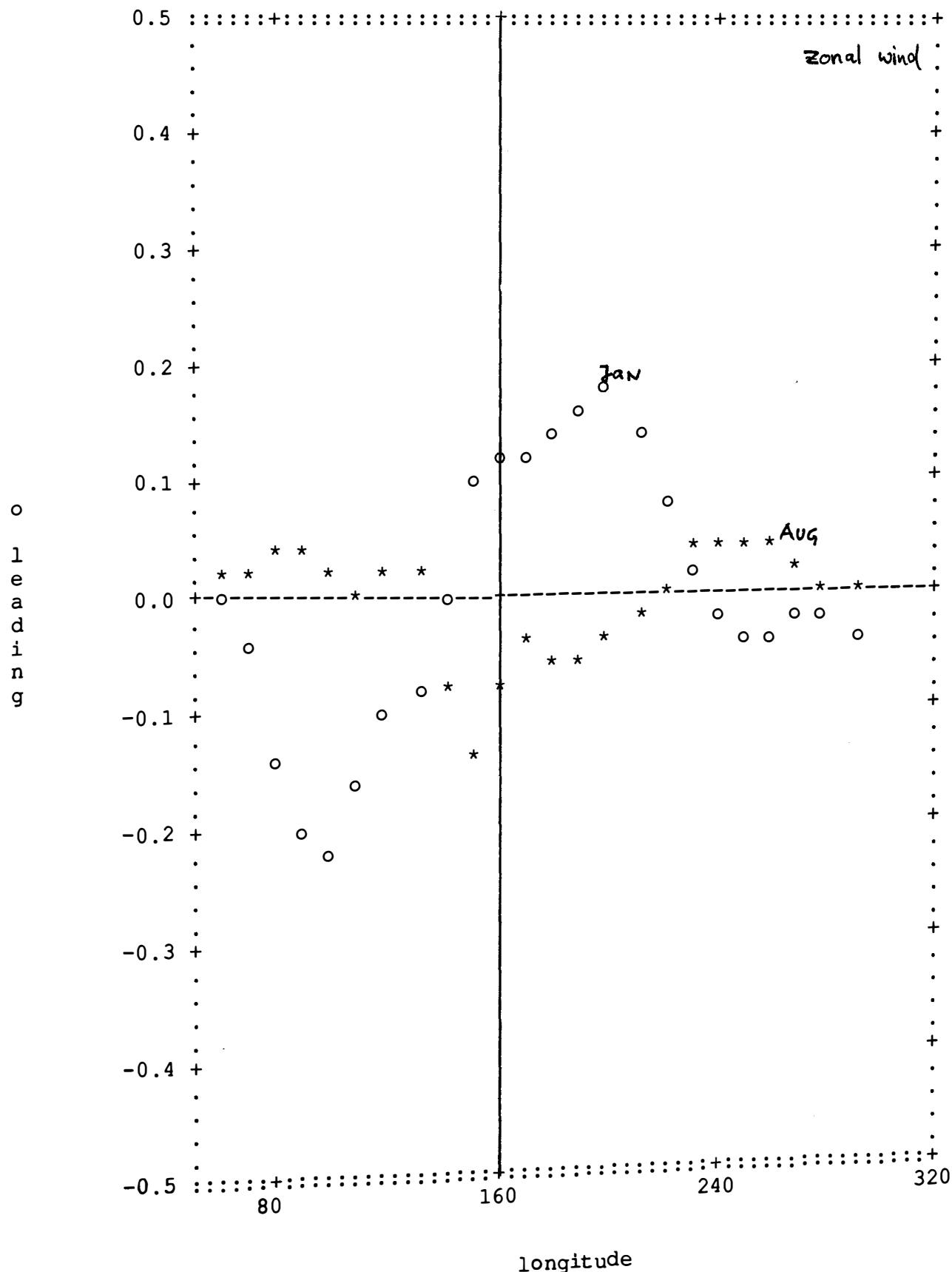


Fig5d2

1

## surface wind: imag Jan / real Aug



data are smoothed

## Cyclostationary POP analysis Conclusions: Method

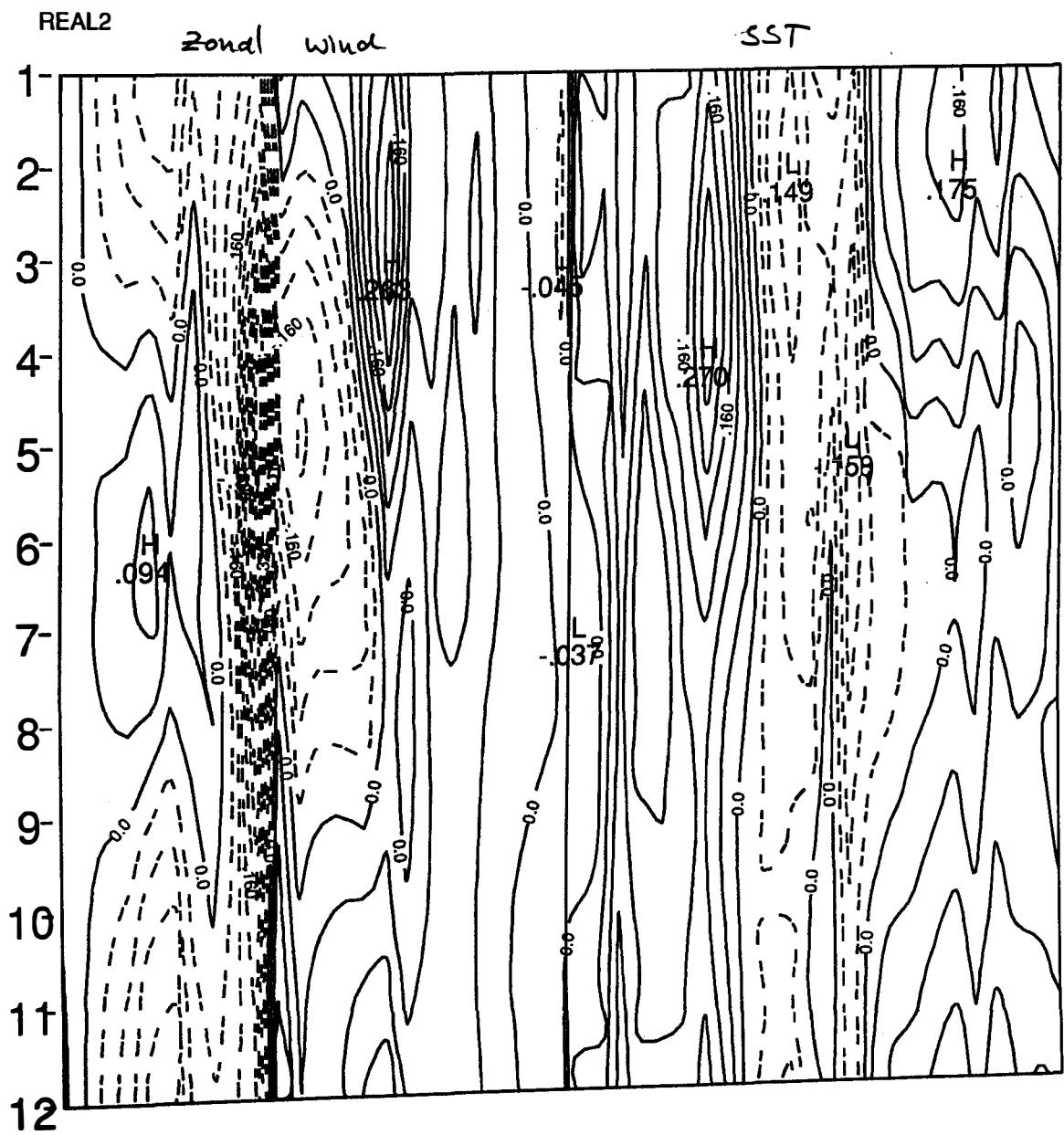
- The CYCLOSTATIONARY POP ANALYSIS is capable to identify seasonally varying memories and patterns.
- The results obtained in synthetical and in real-world examples were consistent with the result of a conventional (stationary) POP analysis.

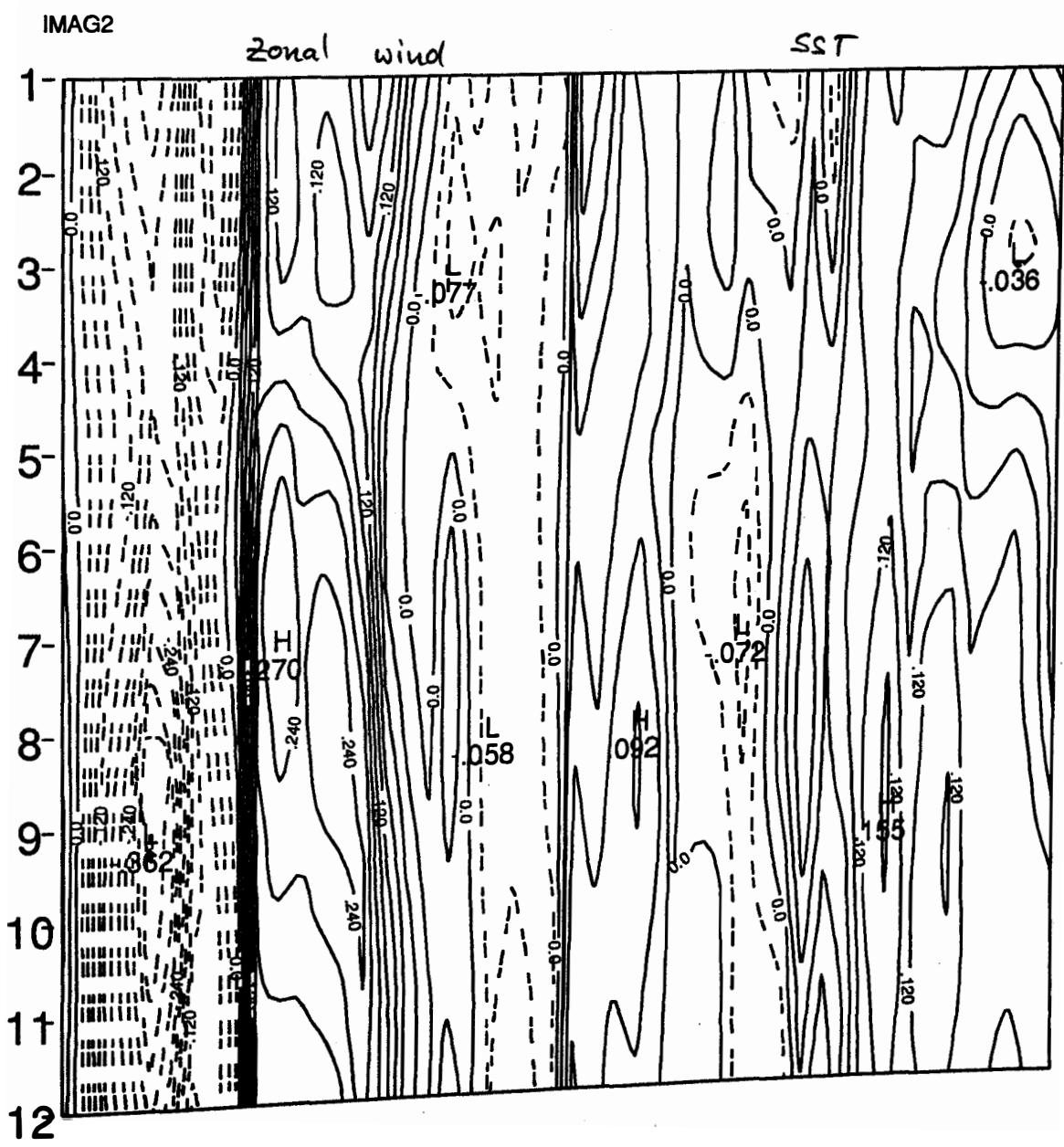
## Cyclostationary POP analysis Conclusions: Physics

- The QBO in the equatorial stratosphere:
  - It is a downward propagating signal, with a period of about two years.
  - The pattern of the QBO is almost independent of the annual cycle.
  - The memory of the QBO is amplifying in March to June, and damped otherwise.
- El Nino / Southern Oscillation
  - ENSO is an oscillatory phenomenon that is eastward propagating in terms of zonal surface wind but standing in terms of SST
  - ENSO is amplified in May to September, and damped in October to March.
  - The patterns vary with the annual cycle.

## Cyclostationary POP analysis Conclusions: Outlook

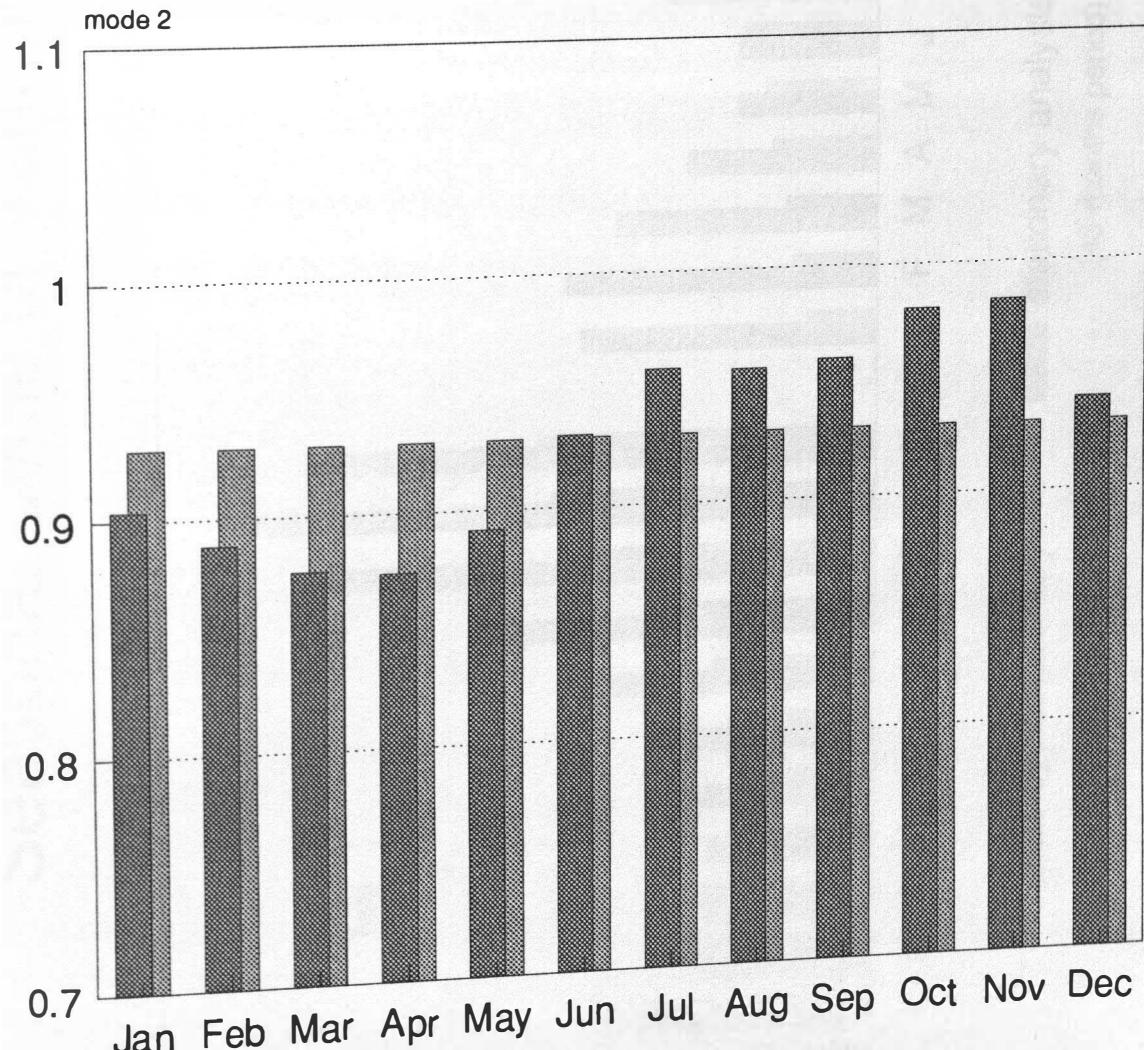
- Use of deterministic cycles not given by time, but by, e.g., the QBO: ENSO
- Use of other deterministic cycles given by time, e.g. diurnal cycle
- Investigation of processes with periods less than the deterministic cycle: the 30-60 day oscillation





# ENSO

## Damping Rates



according to

cyclostationary  
(39 months period)

stationary analysis  
(45 months period)

Fig5d

ENSO

# Seasonal Cycle of Variance

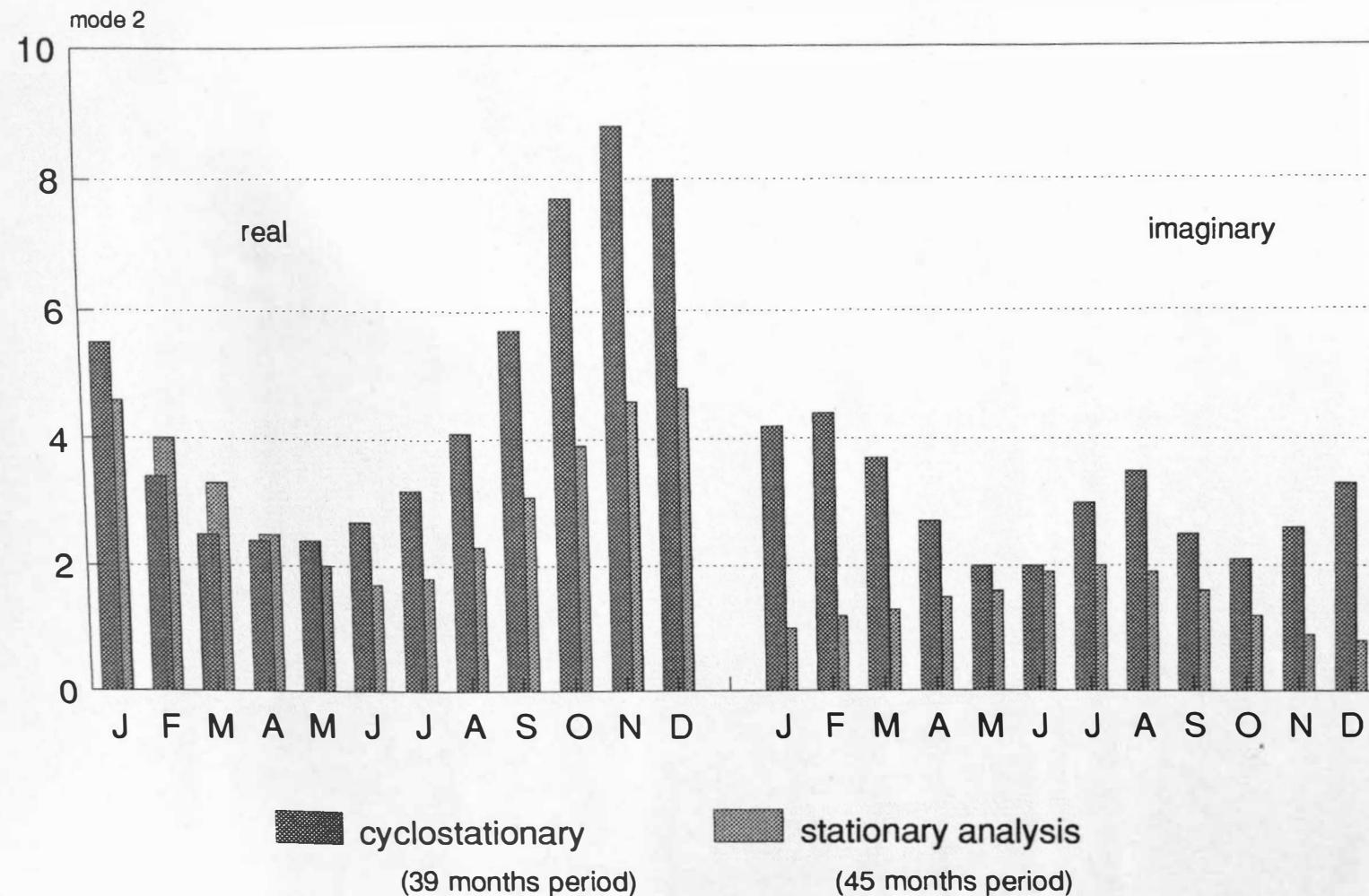


Fig5d4