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Numerical Computation of Optimal Reduction of CO_2-Emissions for a Simplified Climate-Economy Model

Prof. Dr. Hans Joachim Oberle
Institute of Applied Mathematics
University of Hamburg
Bundesstrasse 55
D-20146 Hamburg / Germany

Dr. Hans von Storch
Max-Planck-Institute of Meteorology
Bundesstrasse 55
D-20146 Hamburg / Germany

Dr. Olli Tahvonen
Department of Economics
University of Oulu
The Academy of Finland
SF-90570 Oulu / Finland

Abstract. In this paper a highly simplified model is considered which describes the interaction of anthropogenic climate changes represented by the influences due to the enhanced emission of CO_2 resulting in the increase of the averaged surface air temperature on one side, and the economical effects described by the abatement costs for the reduction of emissions on the other side.

The model is formulated as a linear-quadratic optimal control problem with a compact control region. By applying the standard necessary conditions, a multipoint-boundary-value problem is derived and its numerical solution obtained by multiple-shooting technique is presented. Special attention is paid to the computation of the reachable set of the system and to the dependence of the control structure on the final state prescribed.

In order to smooth a certain irregular behaviour of the solutions near the end of the arbitrarilly fixed time-interval, an additional monotonicity constraint for the CO_2-concentration is introduced. Solutions of this extended optimal control problem are presented too and they are compared with the former solutions.

Key Words. Climate model, optimal control, control constraints, minimum principle, numerical method, reachable set.

AMS (MOS) subject classifications. 34B10, 49K15, 86A10, 90A16.

1. Introduction

In recent years there has been a growing interest in modelling the global climate changes which are due to the man–made increase in the atmospheric concentration of greenhouse gases (cf. IPCC–report [5], Sreenath [11], Tahvonen et al. [14]). The main effects due to the enhanced emissions are an increase in the global mean temperature of the Earth and a rise in the global mean sea level. Under a "business as usual" scenario, the IPCC group expects by the end of the next century anthropogenic CO_2–emission of about 16 GtC [1], an increase of about 3 K in the global mean temperature, and a rise of severel tens of centimeters in the global mean sea level.

Due to this situation, it is of interest to consider realistic models of different complexity for these quantities which allow to propose strategies for the reduction of emissions and to assess the effects of such strategies with respect to the climate changes and to the economy.

In this paper, as a first step a very simplified model due to Tahvonen et al. [14] is considered, which describes the interaction of climate changes and economy. The model is constructed in the form of an optimal control problem with two state variables, representing the globally averaged tropospheric CO_2–concentration and the globally averaged near surface air temperature. The control variable represents the reduction of the CO_2–emissions related to the "business as usual level" of the IPCC study. The simplifications of the model are characterized by the limitation to:

– just only one greenhouse gas, namely CO_2,

– to globally averaged quantities with just one memory term for each quantity,

– and to a linear model for the behaviour of the system in time.

The aim of the model is to determine reduction strategies for the CO_2–emissions such that a certain prescribed final situation with respect to the state variables can be reached within a time–period of hundred years. Further, the reduction strategy determined is an optimal one in the sense that with this strategy the total cost of reduction (abatement costs) is minimized.

In this paper we do not take into account the so–called adaptation costs, i.e. the costs which are due to the adaptation of the world economy to an increased global mean temperature. An estimation and a modelization of these costs would be very uncertain (cf. [14]).

2. A Model for Reduction of CO_2–Emissions

In this section we describe the model used for the interaction of climate and economy introduced by Tahvonen et al. [14].

The dynamic system is given for two state–variables

[1]The unit GtC means gigatons carbon; giga = 10^9

$C(t)$: globally averaged tropospheric concentration of carbon dioxide at time t,
$T(t)$: globally averaged near–surface air–temperature at time t.

Both quantities are given as deviations from their preindustrial values round about the year 1860. Thus, for our time the following initial data are appropriate:

$$C(0) = 73\,\text{ppm}\,, \qquad T(0) = 0.7\,\text{K}\,. \tag{2.1}$$

The time behaviour of these quantities is described by the following linear system of ordinary differential equations

$$\begin{aligned} \dot{C}(t) &= \beta\, E(t) - \sigma\, C(t) \\ \dot{T}(t) &= \mu\, C(t) - \alpha\, T(t)\,. \end{aligned} \tag{2.2}$$

The driving force $E(t)$ is the annual anthropogenic carbon dioxide emission, measured in gigatons carbon dioxide pro year.

As reference behaviour of $E(t)$ we take the IPCC–prediction for the "business–as–usual" scenario, which we approximate by a linear function in time

$$\begin{aligned} E_b(t) &= E_0 + Q\, t \\ E_0 &= 6.7\ \text{GtC/a} \\ Q &= 0.143\ \text{GtC/a}^2\,. \end{aligned} \tag{2.3}$$

Now, in equation (2.2) the emission $E(t)$ is substituted by

$$E(t) = E_b(t) \cdot (1 - R(t))\,, \tag{2.4}$$

where $R(t) \in [0,1]$ denotes the rate of abatement from the uncontrolled emission $E_b(t)$.

The parameters α, β, σ, and μ are assumed to be constant. Their values are empirically determined by fitting the observed record of concentrations and temperatures during the period 1860 – 1985 to the model (2.2) (cf. Maier–Reimer, Hasselmann [6], Marland [7], and Tahvonen et.al. [14]):

$$\begin{aligned} \alpha &= 0.03\ \text{a}^{-1} \\ \sigma &= 0.018\ \text{a}^{-1} \\ \beta &= 0.47\ \text{ppm/GtC} \\ \mu &= 0.45_{10^{-3}}\ \text{K/GtC/a}\,. \end{aligned} \tag{2.5}$$

In Fig. 1 the solutions of the initial–value–problem (2.1), (2.2) are plotted along a time–interval of $t_f = 100\,\text{a}$ for different choices of a constant control function $R(t)$:

$$R(t) = 0.0,\ 0.25,\ 0.5,\ 0.75, \text{and}\ 1.0\,.$$

Further the solution of (2.1), 2.2) is plotted for a time–variant control function $R(t)$ which is chosen such that the CO_2–emission is kept fixed $E(t) \equiv E_0$.

In Fig. 2 the same trajectories are shown in the state plane (C,T). The dashed line indicates those state variables (C,T) which are stationary with respect to the temperature, i.e. $\dot{T} = 0$. The arrows indicate the motion of the state with respect to time $0 \leq t \leq 100$.

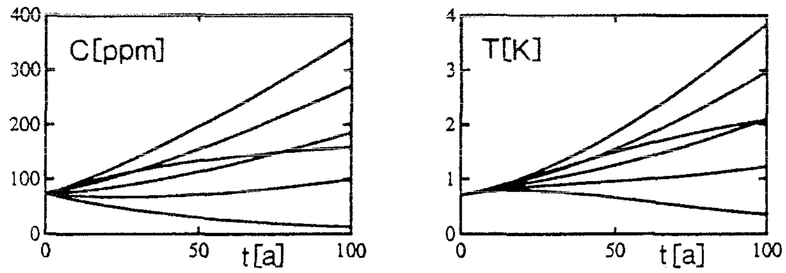

Fig. 1. Time behaviour of the CO_2–concentration and the temperature for different abatement strategies $R(t) = 0.0, 0.25, 0.5, 0.75$, and 1.0 and for $E(t) \equiv E_0$.

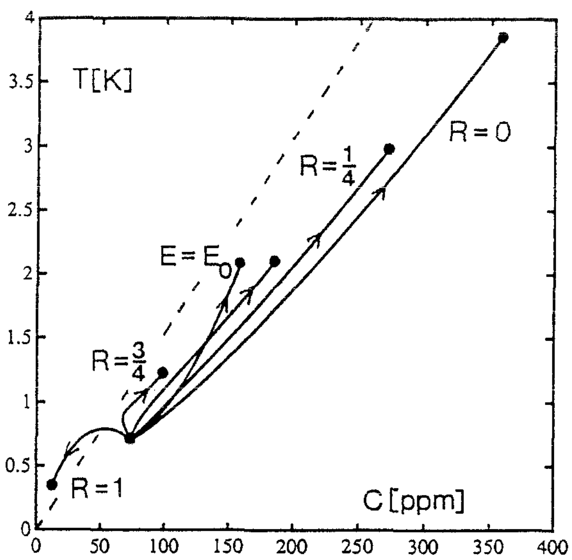

Fig. 2. Trajectories in the state plane (C, T) corresponding to different choices of the control function $R(t)$ as indicated in Fig.1.

3. Statement of the Problem

We consider the model of Section 2 as an optimal control problem with the rate of abatement $R(t)$ as control variable. The aim is to determine $R(t)$, $0 \le t \le t_f$, within a finite time region of $t_f = 100\,\text{a}$ such that a prescribed final state (C_f, T_f) is achieved with a minimum amount of abatement costs.

These costs are measured by the functional

$$I[R] = \int_0^{t_f} A(R)\, e^{(r-\delta)\,t}\, dt. \tag{3.1}$$

Here, r is the rate of growth of net output, and δ the rate of discount. $A(R)$ describes the costs of abatement. In this paper we use a simple quadratic ansatz:

$$A(R) = R^2 , \; r = 0.02 , \; \delta = 0.03 . \tag{3.2}$$

The constraints of the optimal control problem are the equations of motion (2.2), the initial conditions (2.1), the prescribed boundary conditions

$$C(t_f) = C_f , \; T(t_f) = T_f , \tag{3.3}$$

and the control constraints

$$0 \leq R(t) \leq 1 . \tag{3.4}$$

Before we apply the necessary conditons of optimal control theory, it seems to be useful to determine those final states (C_f, T_f) which can be achieved from $(C(0), T(0))$ using an admissible control function, i.e. $R(t)$ is measurable and satisfies (3.4). The set of these final states is called the **reachable set** or the **set of attainability**.

It is well-known from optimal control theory (cf. Strauss[13], Halkin [4]) that each point of the reachable set can be achieved using only control functions with values on the boundary of the control region, so-called bang-bang controls. Therefore, it is obvious that the boundary of the reachable set consists of those final states which are reached by bang-bang controls with just one switching point $\tau \in [0, 100]$, i.e.

$$R_\tau(t) = \begin{cases} 0 , & 0 \leq t \leq \tau \\ 1 , & \tau < t \leq 1 \end{cases} \tag{3.5}$$

or vice versa

$$\tilde{R}_\tau(t) = \begin{cases} 1 , & 0 \leq t \leq \tau \\ 0 , & \tau < t \leq 1 . \end{cases} \tag{3.6}$$

Using these control functions one obtains points of the boundary of the reachable set by solving the initial-value problem (2.1), (2.2) numerically. And, by variation of the parameter $\tau \in [0, 100]$, one obtains the whole boundary of the reachable set.

In Fig. 3 some of these trajectories are shown together with the boundary of the reachable set.

Note again, that precisely the points in the "ellipse" can be achieved with a suitable reduction strategy. So, for instance, the preindustrial situation (corresponding to the origin) cannot be reached within the next hundert years by any kind of reduction strategy.

4. The Necessary Conditions

In this section we apply the necessary conditions of optimal control theory to the CO_2-emission model in order to build up a multipoint boundary value problem with switching conditions for the state- and adjoint variables of the control problem. The notation used

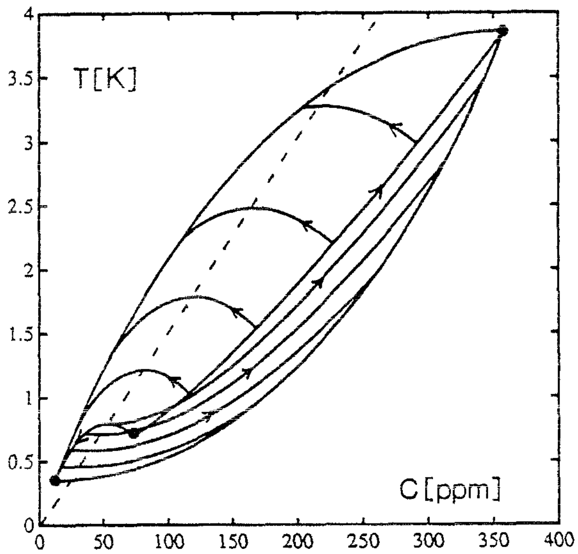

Fig. 3. Reachable set and trajectories with bang–bang control.

is taken from Bryson, Ho [1]. Note, that the problem is nonlinear with respect to the control and, therefore, the necessary conditions for constrained optimal control problems (state–constraint of order zero, cf. Maurer[10]) have to be applied.

Let λ_C, λ_T denote the adjoint variables with respect to C and T, respectively. Then, the Hamiltonian of the problem is given by

$$H \;=\; A(R)\, e^{(r-\delta)\,t} \;+\; \lambda_C \,\{\beta\, E_b(t)\,(1-R) \,-\, \sigma\, C\} \;+\; \lambda_T\, \{\mu\, C \,-\, \alpha\, T\}\,. \qquad (4.1)$$

Here, the classical notation of the Hamiltonian is used instead of the so–called current value Hamiltonian (cf. Feichtinger, Hartl [3], Tahvonen et al. [14]) due to the nonautonomous state equations. Both Hamiltonions differ only by a factor $e^{(r-\delta)\,t}$.

From (4.1) we obtain the following adjoint differential equations:

$$\begin{aligned} \dot\lambda_C &= \sigma\,\lambda_C \,-\, \mu\,\lambda_T \\ \dot\lambda_T &= \alpha\,\lambda_T\,. \end{aligned} \qquad (4.2)$$

The optimal control function is characterized by the minimum principle.

Therefore, for optimal control functions in the interior of the control region we have the following necessary condition

$$\frac{d\,A}{d\,R}\, e^{(r-\delta)\,t} \;=\; \beta\, E_b(t)\, \lambda_C\,. \qquad (4.3)$$

Especially for (3.2) we observe that the Hamiltonian has a unique minimum with respect to the control R which is given by:

$$R_{\text{free}} = \frac{\beta}{2} E_b(t)\, e^{(\delta-r)\,t}\, \lambda_C \ , \tag{4.4}$$

thus, the minimum principle yields the following optimal control law

$$R(t) = \begin{cases} 0 & , \text{ if } R_{\text{free}}(t) < 0 \\ R_{\text{free}}(t) & , \text{ if } 0 \leq R_{\text{free}}(t) \leq 1 \\ 1 & , \text{ if } R_{\text{free}}(t) > 1 \ . \end{cases} \tag{4.5}$$

Altogether, the boundary–value problem is given by the state– and adjoint equations (2.2), (4.2), where the control law is substituted according to (4.4) and (4.5). The corresponding boundary conditions are given by the relations (2.1) and (3.3).

5. Numerical Solution

The numerical solutions of the boundary value problem described in Section 4 are obtained by means of the multiple shooting code BNDSCO (cf. Bulirsch [2], Stoer, Bulirsch [12], Oberle [9], [10]).

For the application of this method to our boundary value problem the switching structure of the solution, i.e. the number and the relative position of the subarcs with different control strategies in Eq. (4.4) has to be estimated a priori. If $\tau_1, \ldots, \tau_s$ denote the junction points (or switching points) between subarcs with different control laws, the following **switching conditions** have to be satisfied:

$$R_{\text{free}}(\tau_k) = 0 \qquad \text{or} \qquad R_{\text{free}}(\tau_k) = 1 \qquad (k = 1, \ldots, s) \ . \tag{5.1}$$

Numerically the switching points are treated as unknown parameters of the boundary value problem which are determined together with the state– and adjoint variables at the multiple shooting nodes by means of the damped Newton method such that Eq. (5.1) is satisfied.

Often a reliable estimation of the switching structure of the solution can be found by inspection of the totally free control, i.e. one solves the two–point boundary value problem (2.1), (2.2), (3.3), and (4.2) with $R \equiv R_{\text{free}}$ neglecting the constraints (3.4). Due to the linearity of this auxiliary boundary value problem the solution can be determined numerically by multiple shooting technique within one or two iteration steps nearly independent of the initial data. Now, one determines those subarcs where the totally free control violates the constraints.

Of course, this way of estimating the switching structure may fail for critical values of the final data C_f and T_f. Therefore it is worthy to note that the boundaries of the regions of all final data (C_f, T_f) whose corresponding optimal control history has a certain prescribed control structure can be determined explicitly by solving suitable parameter–dependend boundary value problems with switching conditions.

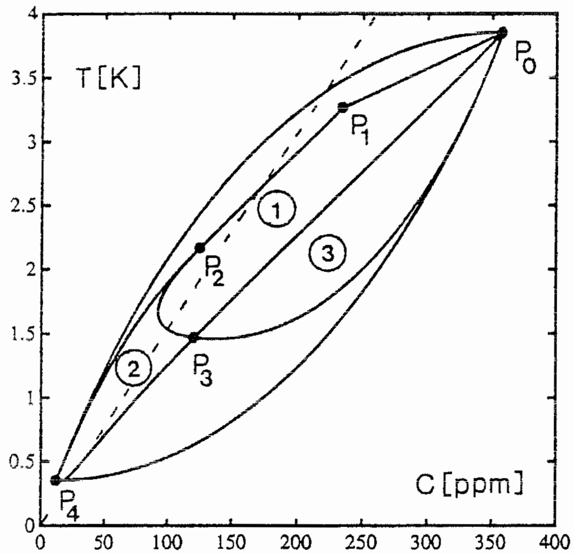

Fig. 4. Reachable set and regions of different control structure.

Three of these regions of different control structures are shown in Fig. 4.

The region ①, characterized by the boundary P_0, P_1, P_2, P_3, P_0, is the region of those final values (C_f, T_f) for which the optimal control is totally free, i.e. the restrictions (3.4) are not active. The boundary of this region is characterized by the condition that the totally free control has just one isolated contact point with the boundary of the control region (or two contact points in the edges). The different subarcs of the boundary are characterized by the conditions given in Table 1.

For the numerical computation one substitutes one of the two boundary conditions (3.3) by the corresponding boundary or switching condition given in Table 1 and varies the other boundary value C_f or T_f, respectively.

The region ② marked in Fig. 4 indicates those final states which are achieved by an optimal control function of the structure

$$R_{\text{free}} \quad - \quad R_{\max} = 1 \quad - \quad R_{\text{free}} \, .$$

Here, the upper boundary $P_2 P_3$ is indicated by the switching structure : $R_{\text{free}} - R_{\max}$ and the additional boundary condition $R_{\text{free}}(t_f) \; = \; 1$. The lower boundary $P_3 P_4$ is indicated by the switching structure : $R_{\text{free}} \; - \; R_{\max} \; - \; R_{\text{free}}$ and the additional boundary condition $R_{\text{free}}(t_f) \; = \; 0$.

Finally, the region ③ marked in Fig. 4 indicates the final states for which the corresponding optimal control function has the structure

$$R_{\text{free}} \quad - \quad R_{\min} = 0 \, .$$

Table 1. Boundary of the control region ①

boundary arc	characterizing condition
$P_0\,P_1$	$R_{\text{free}}(0) = 0$
$P_1\,P_2$	$R_{\text{free}}(t_f) = 1$
$P_2\,P_3$	$R_{\text{free}}(\tau) = 1 ,\ 0 < \tau < 1$
$P_3\,P_0$	$R_{\text{free}}(t_f) = 0$

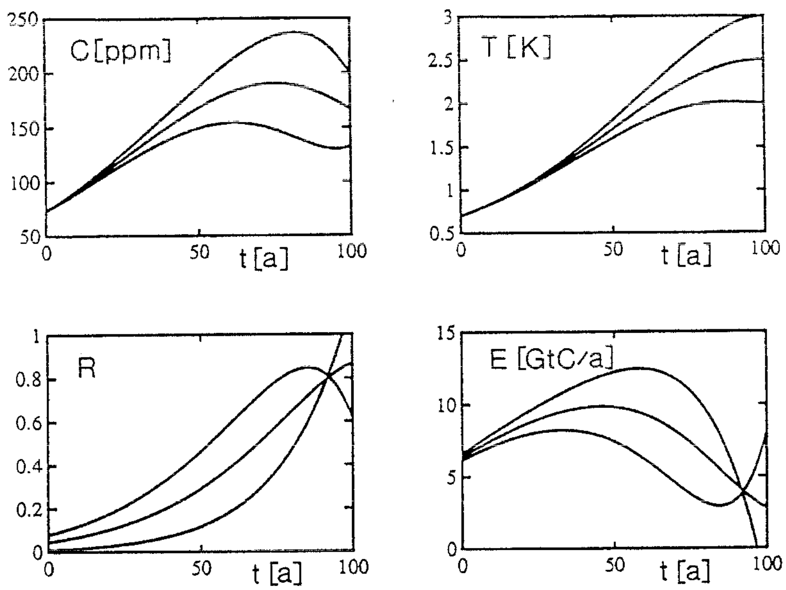

Fig. 5. State- and control histories for the final data $T_f = 3, 2.5, 2$.

The lower boundary $P_3\,P_0$ of this region is characterized by the control structure above together with a switching condition $R_{\text{free}}(\tau) = 1$ indicating that the free control has an isolated contact point with the upper constraint of the control region.

For the remaining parts of the reachable set the corresponding optimal control histories contain nontrivial subarcs on the upper and lower constraints of the control region.

We do not want to stress these regions of different control structures in detail because the final states of practical interest are those which are near the stationary points for the final temperature (dashed line in Fig. 4).

In Fig. 5 the state histories $C(t)$, $T(t)$, the optimal control functions $R(t)$, and the corresponding time history for the emissions $E(t)$ are shown for the prescibed final data $T_f = 3$, 2.5, and 2.

In Fig. 6 the solution histories are shown for $T_f = 1.5$, 1, and 0.7 .

In each case C_f is chosen such that the final temperature is stationary, i.e. $\frac{dT}{dt}(t_f) = 0$. According to the position of the final state in the reachable set (cf. Fig. 4) the optimal control history contains a subarc with maximum reduction for $T_f = 3$, 1.5, 1, and 0.7 . For $T_f = 2.5$ and $T_f = 2$ the optimal control is totally free.

In the following Table 2 the corresponding values of the performance measure $I(R)$, i.e. the total amount of abatement costs (the unit is about 10^{12} dollars, c.f. [14]) are given.

Table 2. Minimal abatement costs for different values of the final state

$T(t_f)$	$C(t_f)$	$I(R_{\text{opt}})$
3.0	200.00	5.9071
2.5	166.67	8.9804
2.0	133.33	14.320
1.5	100.00	21.919
1.0	66.67	32.393
0.7	46.67	41.687

6. Regularization of the Asymptotic Behaviour

A serious drawback of the treatment of finite time horizonts is the somehow irregular behaviour of control and state functions near the end of the time interval considered. This is due to the fact that the formulation of the optimal control problem used in Section 3 does not take into account the behaviour of the state for times after the artificially prescribed final time t_f.

In Figs. 5, 6 examples are shown for which the emissions grow again at the end of the time period (hundred years) considered. Therefore, using this (optimal) control function, the CO_2-concentration would grow after the final time too, if one assumes that the reduction strategy varies continuously. Thus, the stationarity of the state in $t = t_f$ cannot be maintained for $t > t_f$, independently of the further reduction management.

In order to overcome this irregular behaviour in [14] a certain penalty term is added to the performance index which minimizes a weighted sum of the two state variables in the end point of the time interval. However, this method has the drawback that it is not clear how to choose the weight parameters appropriately.

Also, the consideration of an infinite planing horizon does not necessarily overcome this difficulty, because we would like to achieve a stable situation (stationarity) with respect

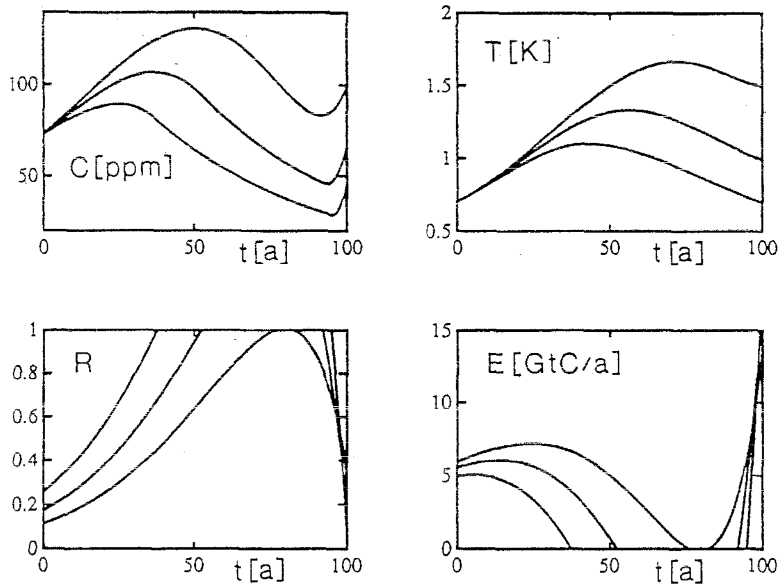

Fig. 6. State- and control histories for the final data $T_f = 1.5, 1, 0.7$.

to the emissions, the CO_2–concentration and the increased temperature on a low level within a reasonable **finite** time–interval.

Therefore, in this paper we propose an alternative way to overcome the irregular behaviour near the end of the planing interval. To this end we add the following inequality constraint (monotonicity constraint)

$$\dot{C}(t) \leq 0 , \qquad \text{for all } t \geq t_1 , \tag{6.1}$$

to the optimal control problem. Here, t_1 denotes the first stationary point of the concentration $C(t)$. With this restriction the CO_2–concentration is not allowed to increase after the first time we have achieved the stationarity of $C(t)$.

Explicitly, (6.1) is an additional inequality constraint to the control and can be reformulated as follows:

$$R(t) \geq 1 - \frac{\sigma\, C(t)}{\beta\, E_b(t)} , \qquad \text{for all } t \geq t_1 . \tag{6.2}$$

In this form the constraint (6.1) can be handled in the same way as the more simple control constraints (3.4). More precisely, if (6.2) is active on a certain subinterval $[\tau_1, \tau_2]$, one introduces the corresponding switching points τ_1, τ_2 as additional variables in the boundary–value problem, one uses the boundary control

$$R_{\text{bound}}(t) := 1 - \frac{\sigma\, C(t)}{\beta\, E_b(t)} \tag{6.3}$$

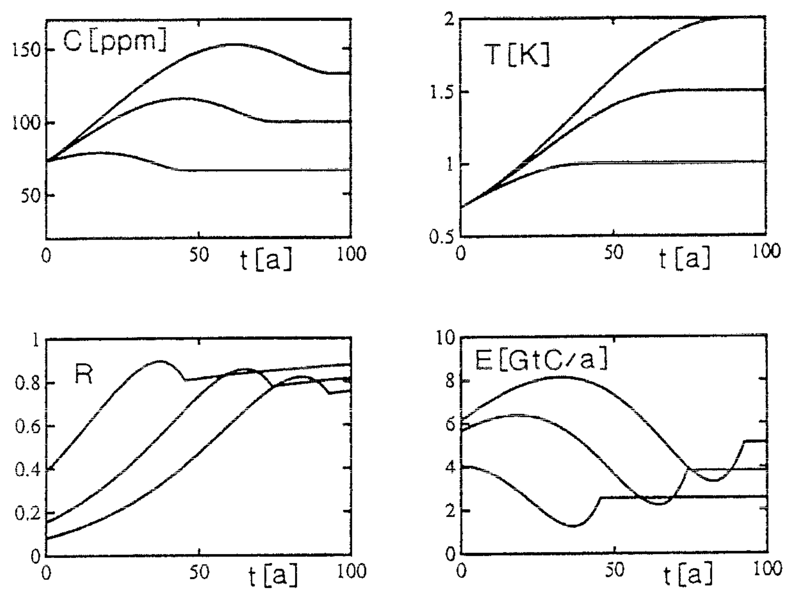

Fig. 7. State- and control histories for the final data $T_f = 2, 1.5, 1$.

on this subinterval, and demands (switching conditions) that the boundary control and the free control coincide at these switching points.

In Fig. 7 the optimal time histories of the state and control variables for this problem are shown. For the final temperature we chose again $T_f = 2$, 1.5, and 1. C_f is chosen such that the final temperature T_f is stationary.

The behaviour of these trajectories can be compared directly with the solution behaviour shown in Fig. 5–6 for the problem without monotonicity constraint. One observes that due to (6.2) the basic constraint (3.4) is not longer active (for these final data however) and that the stationarity with respect to both state variables is reached already at the corresponding switching time.

Of course the demand on the reduction function $R(t)$ is more severe for the constrained problem.

In Table 3 the corresponding values of the optimal performance measure $I(R)$ are listed. Compared with the corresponding values in Table 2 one observes an only mild increase in the reduction costs.

7. Conclusions

In this paper, we have presented numerical solutions of a constrained linear–quadratic optimal control problem which describes the interaction of climate changes and economy.

Table 3. Minimal abatement costs under additional monotonicity constraint

$T(t_f)$	$C(t_f)$	$I(R^*_{\text{opt}})$
2.0	133.33	14.339
1.5	100.00	22.727
1.0	66.67	37.579

The two-dimensional state represents the globally averaged CO_2-concentration and the globally averaged increase of temperature. The aim was to determine optimal reduction strategies for the CO_2-emissions such that the present state is transfered to a desired (stationary) final state within a finite time-interval in such way that the total abatement costs are minimized.

The dependence of the control structure on the prescribed final data are investigated and the reachable set and the regions of different control structure are computed by solving parameter-depending multipoint boundary value problem.

In order to smooth the irregular behaviour of the solutions near the end of the arbitrarilly fixed time-interval, an additional monotonicity constraint for the CO_2-concentration is introduced. Solutions of this extended optimal control problem are presented and they are compared with the former solutions.

The authors are well aware of the simplicity of the model considered, which may allow only to predict a reasonable tendency of the efforts necessary to solve the practical problem. Further investigation are necessary to establish more realistic models which may take into account the effects of different greenhouse gases with different memory terms and more refined (possibly nonlinear) equations of motion.

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