Construction of Optimal Numerical Filters Fitted for Noise Damping in Numerical Simulation Models

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Abstract

A method is presented to get optimal numerical filters that does not affect the convergence-properties of the difference-approximation. Results are shown and compared with the properties of the correxponding Shapiro filters.

Zusammenfassung: Konstruktion optimaler numerischer Filter zur Lärmdämpfung in numerischen Simulationsmodellen

Ein Verfahren zur Gewinnung optimaler numerischer Filter, die die Stabilität und Konsistenz des Differenzenverfahrens nicht beeinflussen, wird dargestellt. Es werden verschiedene auf diese Weise konstruierte Filter angegeben und mit den entsprechenden von Shapiro verglichen.

Résumé: Construction de filtres numériques optimaux pour l'amortissement du bruit dans les modèles de simulation numérique

On présente une méthode pour obtenir des filtres numériques optimaux qui n'affectent pas les proprieétés de convergence de l'approximation aux différences finies.

On compare divers filtres obtenus de cette façon avec les filtres de Shapiro correspondants.

1 Introduction

In simulating the general circulation of the atmosphere through finite difference techniques it is necessary to damp out small-scale pertubations with scales near the lower boundary of the models resolution. Otherwise the meteorological fields will eventually be destroyed by nonlinear instability, which is due to the aliasing effect)which can partly be avoided by a suitable difference scheme; ARAKAWA, 1970), by imbalances due to small-scale forcing terms (convection, mountains ...) and boundary effects. (PHILLIPS, 1959; ROECKNER, 1976).

The necessity of filtering short waves also appears in the finite element method (CULLEN, 1976), the pseudospectral method (MERILESS, 1974) and the spectral method (BATCHELOR, 1969).

There are different procedures for filtering:

(1) Adding *diffusion-terms* to the equations. CULLEN (1976) and SHAPIRO (1971) have listed several possibilities;

(2) Applying the *chopping-method*, which consists of an expansion of meteorological fields in orthogonal functions with a recombination omitting high wavenumbers. In spherical grid representation a convenient way is the Fourier filter, where a harmonic analysis along latitudinal circles is performed and all wavelengths that are shorter than for instance $4\Delta x$ are disregarded, where Δx is the zonal grid interval. (See PHILLIPS, 1959; WILLIAMSON, 1974; MERILEES, 1974);

(3) *Numerical filtering.* In this method each value of a gridpoint is replaced by an average formed with surrounding gridpoint values (for an exact definition, see Section 2). The first numerical filter in meteorology was presented by SHUMAN (1958) and applied by WALLINGTON (1962) in a numerical weather prediction model.
A special class of numerical filters was introduced by SHAPIRO (1970) and adopted among others by FRANCIS (1975), CULLEN (1976) and MUDRICK (1976).

(4) *Diffusive finite difference schemes* possess a "built in" mechanism of damping short waves. The Lax-Wendroff scheme, the Euler method and the "upstream" differencing should be mentioned in this context (see e.g. KREISS and OLIGER, 1973).

The discussion, which of the four methods is the best, has not yet come to an end. In what follows, suitable numerical filters similar in their characteristics to those described by SHAPIRO (1975), will be presented.

When applying a numerical filter one must ensure that it does not affect the convergence properties of the difference approximation. Therefore, the filter must be a consistent approximation of the identity operator and must be stable, i.e. no wave component may be intensified. A series of such filters were proposed in the past, e.g. by SHUMAN (1958) and SHAPIRO (1971, 1975), they are, however, not 'optimal' in the sense, that the best scale selection with a certain amount of computation is achieved.

The necessary range of scale selectivity depends on the model's difference formulation. If the latter does not exclude aliasing, the filters should damp about the upper third of the wavenumber range (ORSZAG, 1971). When, on the other hand, employing a difference scheme with de facto no aliasing (Arakawa-type schemes) the filter should only affect the very high wavenumbers.

We will describe a method to get 'optimal' filters that are especially suitable for schemes without aliasing and display some results. It is planned to report in a subsequent paper, how the filters work in a general circulation model.

The problem of constructing such filters will be regarded as an optimization problem *with* (linear) constraints. This is a wellknown problem in applied mathematics and can be solved by known methods.

Already formerly the problem of constructing filters was interpreted as an optimization problem without constraints, i.e. as an approximation problem. However, filters obtained in this way were in general not stable as they were not aimed at guaranteeing computational stability in numerical simulation models. The first publication in this field was that of BLECK (1965), who used filters for analysis purposes. Independently of BLECK this idea was pursued in the geophysical literature by GALLI and RANDI (1967) with a subsequent correction by ZELEI (1971).

2 The formulation of the optimization-problem with linear constraints

Definition:

Let the interval $(0.2\,\pi)$ be equally spaced into M pieces with gridlength $\Delta x := \frac{2\pi}{M}$ (M an even number). Let D be the set of realvalued 2π-periodic functions defined on the grid $G := \{j\,\Delta x;\ j \in \mathbb{Z}\}$ and $a_0, \ldots, a_m$ real numbers ($m \geqslant 1$). The map

$$T: D \longrightarrow D$$

defined by

$$(Tf)(x) := a_0 f(x) + \sum_{j=1}^{m} a_j \left[f(x + j\,\Delta x) + f(x - j\,\Delta x)\right]$$

$(x \in G)$ is called a numerical filter of m-th order. The numbers $a_0, \ldots, a_m$ are called the *weights* of the filter T.

The possibility of application results from the following fact: As known every function $f \in D$ can be expanded into a Fourier-sum:

$$f(x) = q_0 + \sum_{n=1}^{M/2} q_n^s \sin(nx) + q_n^c \cos(nx)$$

with $q_{M/2}^s = 0$. For every numerical filter there are numbers $h(n) \in \mathbb{R}$ so that for every function $f \in D$ the following representation is valid:

$$(Tf)(x) = h(0)q_0 + \sum_{n=1}^{M/2} h(n)\,[q_n^s \sin(nx) + q_n^c \cos(nx)]$$

(Note that for all real $n \in \mathbb{R}$ the function

$$g_n: \mathbb{C} \longrightarrow \mathbb{C};\ g_n(x) := e^{inx}$$

is an eigenfunction of T with the eigenvalue

$$h(n) := a_0 + 2 \sum_{j=1}^{m} a_j \cdot \cos\ nj \frac{2\pi}{M} \quad)$$

The function h is called the *response function* of the filter T. Each filter is determined by its response function.

A filter must have some *properties* if it shall be used in a numerical simulation model for the purpose of noise control:

(1) The mean value (wavenumber 0) remains unchanged:

$$h(0) = 1$$

This property may be interpreted as being a consistent approximation of the identiy operator.

(2) None of the wave-components which are resolved by the model (thus the wavenumbers, $0, 1, \ldots, \frac{M}{2}$) is intensified.

$$h(n) \leqslant 1 \qquad \text{for } n = 0, 1, \ldots, \frac{M}{2}$$

This prescription may be defined as a stability property.

(3) The filter causes no phase reversal, i.e. the response function is for all resolved wavenumbers non-negative.

$$h(n) \geqslant 0 \qquad \text{for } n = 0, 1, \ldots \frac{M}{2}$$

(4) The shortest resolvable component vanishes:

$$h\left(\frac{M}{2}\right) = 0$$

(5) The modification of the meteorologically relevant wave range does not exceed a certain prescribed amount.

$$h(n) \geqslant r_n \qquad \text{for } n = 0, 1, \ldots, \frac{M}{2}$$

with some values $r_n \in \mathbb{R}$.

(6) Moreover, all components except the very high wavenumbers remain mostly unchanged, i.e. with some norm $\|\cdot\|$:

$$\| 1 - h \| \overset{!}{=} \min$$

Possible norms are for instance the "l_1-norm"

$$\| 1 - h \|_1 := \sum_n | 1 - h(n) |$$

or the "l_2-norm"

$$\| 1 - h \|_2 := \sqrt{\sum_n (1 - h(n))^2}$$

With the notation

$$\underline{\beta}_n := (\beta_{no}, \ldots, \beta_{nm})$$

where

$$\beta_{ni} := \begin{cases} 1 & \text{if } i = 0 \\ 2 \cos(n i \Delta x) & \text{otherwise} \end{cases}$$

and

$$\underline{a} := (a_0, \ldots, a_m)$$

we get the representation

(*) $\quad h(n) = \underline{\beta}_n \cdot \underline{a}'$

for all n. (The prime denotes the transposition of the column vector $\underline{a}$ into a row vector.)

According to the representation (*) we can reformulate conditions (1) to (6) and comprehend them:

(1′) $\quad \underline{\beta}_0 \cdot \underline{a}' = 1$

(2′) $\quad \underline{\beta}_n \cdot \underline{a}' \leqslant 1 \qquad$ for $n = 1, \ldots, \frac{M}{2} - 1$

(3′) $\quad \underline{\beta}_n \cdot \underline{a}' \geqslant r_n \qquad$ for $n = 1, \ldots, \frac{M}{2} - 1$

(4′) $\quad \underline{\beta}_{M/2} \cdot \underline{a}' = 0$

and

(P′) $\quad - \left(\sum_{n=0}^{M/2} \underline{\beta}_n \right) \cdot \underline{a}' \overset{!}{=} \min$

or

(P″) $\quad \underline{a} \cdot \left(\sum_n \underline{\beta}_n' \cdot \underline{\beta}_n \right) \cdot \underline{a}' - 2 \left(\sum_n \underline{\beta}_n \right) \cdot \underline{a}' \overset{!}{=} \min$

The problem (1′) to (4′) with (P′) or (P″) is a wellknown problem in applied mathematics. It is the so-called *optimization problem.* Conditions ($1_1'$) to (4′) are called the (linear) *constraints*, and the minimizing functional from (P′) or (P″) is called the *target function.*

In the case of (P′) the problem is said to be *linear* for the target-function has the form

$$\underline{c} \cdot \underline{a}'$$

with some vector $\underline{c}$. The case (P″) is called *quadratic,* because there exist a matrix $\underline{A}$ and a vector $\underline{c}$ so that the target-function may be written in the form

$$\underline{a} \cdot \underline{A} \cdot \underline{a}' + \underline{c} \cdot \underline{a}'$$

There are methods to decide whether there exist solutions of the optimization problem (e.g. if $r_n = 1$ for $n = 1, \ldots, \frac{M}{2} - 1$, there is no solution.) and in case of existence how to get them.

For the linear case we have the so-called *simplex-method* which leads to the exact solution after a finite number of operations. Though the problem cannot be solved exactly in the nonlinear case, it can be well approximated. As we shall see later on, the quadratic formulation with (P″) is inferior to the linear one with (P′). (For the topic 'optimization problems' see e.g. COLLATZ and WETTERLING, 1971).
The filters being defined with the solution vector of the optimization problem will be called the *optimal numerical filter* (ONF) of m-th order. It will be denoted by

$$K_l^m$$

in the linear case and by

$$K_q^m$$

in the quadratic case. SHAPIRO's m-th order filter (SHAPIRO, 1975) will be denoted by

$$S^m$$

3 The linear case

In the linear case the lower boundary function r_n was chosen as the corresponding SHAPIRO filter, i.e. that of same order:

$$r_n := S^m(n)$$

In both the linear and the quadratic case $M = 128$ was prescribed. For arbitrary order of m there exists a solution of the linear optimization problem and this one is more scale-selective than the SHAPIRO filter of same order, i.e. we get for all considered wavenumbers n:

$$S^m(n) \leqslant K_l^m(n) \leqslant 1$$

and

$$S^m\left(\frac{M}{2}\right) = K_l^M\left(\frac{M}{2}\right) = 0$$

For $m = 1, 2$ and 3, the maximal difference between K_l^m and S^m is less than 0.1 %. Thus K_l^m and S^m are practically equal for $m = 1, 2$ and 3.
For $m \geqslant 4$ the differences increase however, as can be seen from Table 1 and Figure 1.

The ONFs of order $m \geqslant 8$ are even more scale-selective than SHAPIRO's of order $m + 1$, i.e.

$$S^{m+1}(n) \leqslant K_l^m(n) \leqslant 1 \qquad \text{for } m \geqslant 8$$

For instance, one gets a better result if one works with the 17-points ONF K_l^8 than if one applies the 19-point filter S^9. Furthermore, the application of the 15-point filter K_l^7 yields essentially the same effect as the 17-point filter S^8. K_l^8 is as good as S^{10} (see Figure 2).

In Table 2 the weights of the ONFs of order 4 up to 10 are listed. Though they are computed for $M = 128$, they are applicable for other values of M, too.

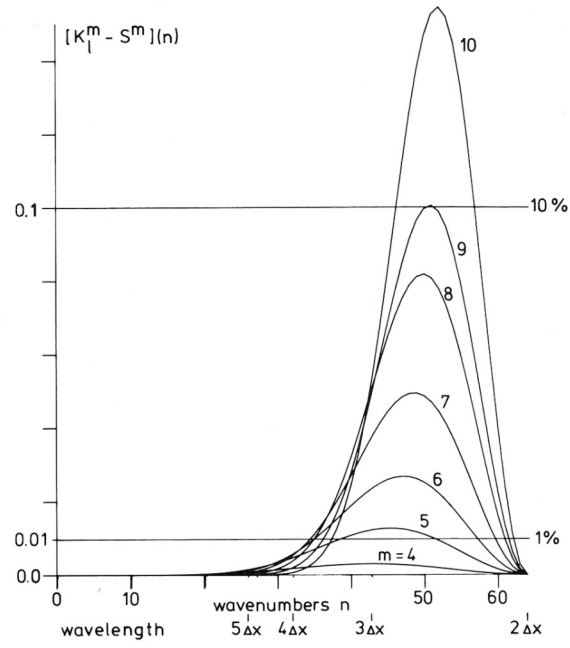
4 The quadratic case

As already mentioned there is no method giving the exact solution of the quadratic problem (1′) to (4′) and (P″). Therefore one has to apply an approximation method. Cryer's method (see ECKHARDT

■ **Table 1** Maximal differences between the response-functions of the ONF and the corresponding Shapiro-filter of order m. (m = 4, ..., 10) (linear formulation).

■ **Tabelle 1** Maximaler Unterschied der Response-Funktionen des ONF und des Shapiro-Filters gleicher Ordnung m. (m = 4, ..., 10) (lineare Formulierung).

m	$\max_n (K_l^m - S^m)(n)$
4	0.3 %
5	1.2 %
6	2.6 %
7	4.9 %
8	8.1 %
9	10.0 %
10	15.4 %



● **Figure 1**
Differences between K_l^m and S^m for m = 4, 5, ..., 10

● **Bild 1**
Differenzen zwischen K_l^m und S^m für m = 4, 5, ..., 10

(1974)) was used. Solving such a nonlinear optimization problem is quite expensive regarding computer time. The essential factor is the number of constraints. Therefore, only every forth wavenumber in the constraints was considered. This simplification had no substantial influence on the solution given by the approximation method.

For the case M = 128 and m = 4, one computation with 10,000 and one with 40,000 iterations were performed. The resulting vectors are listed in Table 3.
In Figure 3 the corresponding response-functions and the one of the 4-th order Shapiro filter are plotted. Our constructed filters seem to be rather good, but that is not really true for the following reason: Because the results given by the approximation method differ a little from the exact solution and do not fulfill the constraints exactly, the corresponding filters do not satisfy the conditions (1′) to (4′). For instance, one has instead of

$$K_q^m(0) = 1$$

only

$$K_q^m(0) = 1 - \epsilon_0$$

with some small number ϵ_0. Now, in a numerical simulation model the filter is frequently applied. The effect on the component with wavenumber 0 after k applications of the filter is:

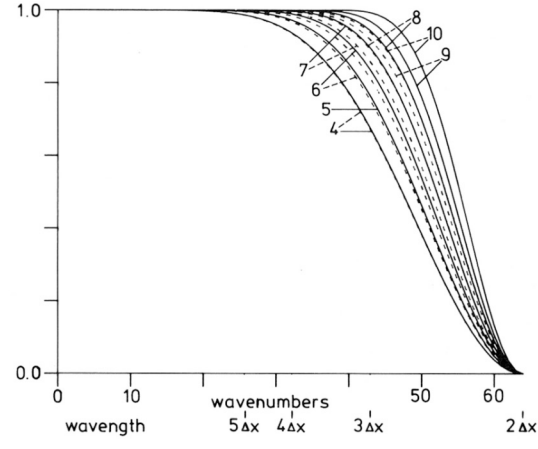
$$[K_q^m(0)]^k = (1 - \epsilon_0)^k \approx 1 - k\epsilon_0$$

■ **Table 2** Weights for the ONFs K_l^m, m = 4, 5, ..., 10.

■ **Tabelle 2** Gewichte der ONFs K_l^m, m = 4, 5, ..., 10.

j	m = 4	5	6	m = 7	8	9
0	0.727 670 918	0.758 103 098	0.782 476 717	0.804 185 144	0.824 375 512	0.838 357 833
1	0.218 304 502	0.202 951 266	0.188 654 145	0.174 490 024	0.160 135 956	0.149 505 138
2	−0.109 816 938	−0.118 353 640	−0.122 139 301	−0.122 851 202	−0.120 973 724	−0.117 993 829
3	0.031 695 497	0.045 911 775	0.057 562 438	0.067 244 896	0.074 875 279	0.078 816 733
4	−0.004 018 520	−0.010 697 909	−0.018 741 086	−0.027 725 733	−0.037 169 310	−0.043 877 157
5		0.001 136 958	0.003 783 416	0.008 130 147	0.014 270 381	0.019 846 086
6			−0.000 357 971	−0.001 514 636	−0.003 981 867	−0.007 005 454
7				0.000 134 932	0.000 718 381	0.001 807 459
8					−0.000 062 854	−0.000 302 475
9						0.000 024 583

j	m = 10
0	0.857 400 094
1	0.134 281 233
2	−0.111 963 274
3	0.082 222 793
4	−0.052 572 277
5	0.028 678 762
6	−0.012 936 578
7	0.004 609 715
8	−0.001 210 538
9	0.000 207 496
10	−0.000 017 378



● **Figure 2** The response-functions for K_l^m (solid lines) and for S^m (dashed lines), m = 4,5 ..., 10

● **Bild 2** Die Response-Funktionen für K_l^m (durchgezogene Linien) und für S^m (gestrichelte Linien) m = 4, 5, ..., 10

If $\epsilon_0 = 0.001$ and k = 240 one gets a deviation of about 0.24 = 24 %! (A simulation over 5 days with filtering every 30 minutes requires 240 filterapplications.)

In Figure 4 the response-function of the 240-times application of K_q^4 (40,000 iterations) and S^4 is plotted. One can see that the 'optimal' filter is useless for our purposes.

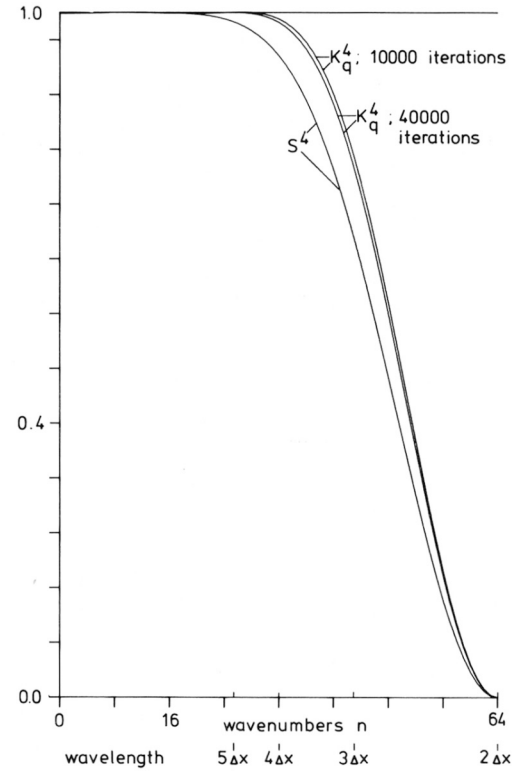
5 Conclusion

It turns out to be a successfull approach to formulate the problem of constructing filters for noise control as an optimization problem with linear constraints. The procedure working with the l_1-norm gives results superior to those known so far. For instance, the optimal 17-point-filter is a good as the 21-point-filter of Shapiro. On the other hand the nonlinear approach using the l_2-norm did not lead to filters appropriate for our purposes.

■ **Table 3** Weights for the ONFs of 4-th order with quadratic formulation.

■ **Tabelle 3** Gewichte für die ONF der Ordnung 4 bei quadratischer Formulierung der Optimierungsaufgabe.

j	10,000 iterations	40,000 iterations
0	0.767 194 620	0.761 804 339
1	0.199 102 750	0.202 080 218
2	−0.122 877 590	−0.121 506 805
3	0.050 550 013	0.047 788 626
4	−0.011 063 128	−0.009 526 418

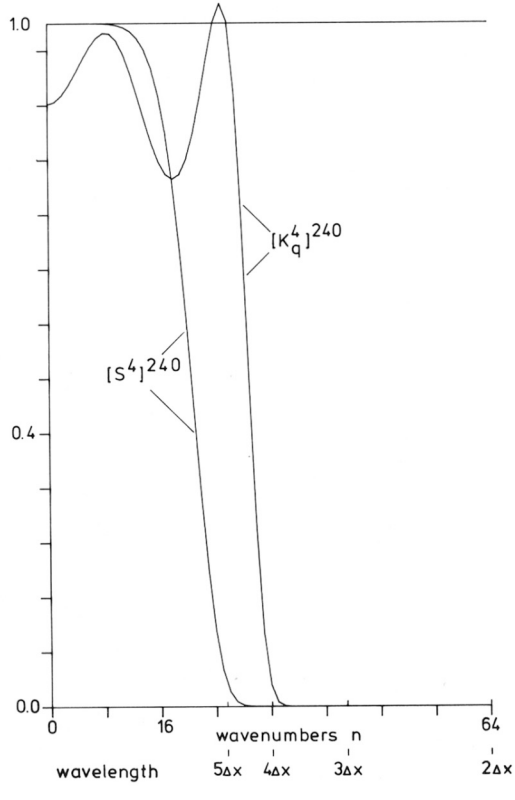


● **Figure 3**

Response-functions for K_q^4 with 10,000 and 40,000 iterations and for S^4

● **Bild 3**

Response-Funktionen für K_q^4 (berechnet mit 10 000 und 40 000 Iterationen) und für S^4



● **Figure 4**

The response-functions for $(K_q^m)^{240}$ (40,000 iterations) and for $(S^m)^{240}$

● **Bild 4**

Die Response-Funktionen für $(K_q^m)^{240}$ (40 000 Iterationen) und für $(S^m)^{240}$

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